ELE 724
Chapter 4: Delta-Sigma Modulators

Lecture Review Notes - Part 4

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1st-Order Delta-Sigma Modulators

- Signal passes through 1st-order $\Delta \Sigma$ modulators without attenuation while quantization noise is first-order shaped with noise transfer function $NTF = 1 - z^{-1}$. Under the condition $\omega T_s \ll 1$, i.e., $\omega \ll \omega_s$ where $\omega_s$ is the clock frequency, $|NTF(j\omega)| \approx \omega T_s$, i.e., quantization noise at frequency $\omega \ll \omega_s$ is pushed to higher frequencies. As a result, a higher SNR at frequencies $\omega \ll \omega_s$, i.e., in the signal band, can be achieved.

- In time-domain, the noise at the output of the quantizer of 1st-order $\Delta \Sigma$ modulators is the difference of the two consecutive samples of quantization noise, i.e., $e(n) - e(n-1)$ (In frequency domain, it is $(1 - z^{-1})E(z)$). Since fundamentally quantization noise is not white but rather input signal dependent, $e(n)$ and $e(n-1)$ are correlated. $e(n) - e(n-1)$ largely removes the correlated portion of the quantization noise. As a result, lower quantization noise at the output of the modulator (at frequency $\omega \ll \omega_s$) is obtained.\(^1\)

\(^1\)An analogy of this is a differential amplifier. If we consider the correlated portion of quantization noise as the common-mode portion of the input of the amplifier while the uncorrelated portion of quantization noise as the differential-mode portion of the input of the amplifier, then the common-mode portion of the input is removed via the differential operation of the amplifier.

\(^2\)The operation of the subtraction of two consecutive quantization noise samples is a powerful technique called double-sampling widely used in eliminating the deterministic errors of systems such as the input offset voltage of amplifiers, the fixed pattern noise of CMOS image sensors, and non-white noise such as flicker noise of MOSFETs.
The excessive quantization noise at high frequencies can be removed using a digital filter immediately following the quantizer called decimation filter. Decimation (down-sampling) is needed as the output of the modulator contains (i) input signal, (ii) sampling clock, and (iii) displaced excessive quantization noise. Decimation filters perform 3 tasks:

1. Filter out sampling clock
2. Filter out displaced excessive quantization
3. Pick up the input signal.

The down-sampling ratio or sampling frequency of decimation filters is determined by the bandwidth of the input signal as per Nyquist theorem, which is low. As a result, under-sampling of quantization noise occurs. Large noise might exist at the output of the decimation filters.
2nd-Order Delta-Sigma Modulators

Signal passes through 2nd-order \( \Delta \Sigma \) modulators without attenuation while quantization noise is 2nd-order shaped with noise transfer function \( NTF = (1 - z^{-1})^2 \). Under the condition \( \omega T_s \ll 1 \), i.e., \( \omega \ll \omega_s \) where \( \omega_s \) is the clock frequency, \( |NTF(j\omega)| \approx (\omega T_s)^2 \), i.e., quantization noise at frequency \( \omega \ll \omega_s \) is pushed to higher frequencies at a rate that is 20 dB higher as compared with that of 1st-order \( \Delta \Sigma \) modulators. As a result, a higher SNR at frequencies \( \omega \ll \omega_s \), i.e., in the signal band, can be achieved.

In time-domain, the noise at the output of the quantizer of 2nd-order \( \Delta \Sigma \) modulators is the difference of the difference of the consecutive samples of quantization noise, i.e., \([e(n) - e(n-1)] - [e(n-1) - e(n-2)]\) (In frequency domain, it is \((1 - z^{-1})^2E(z))\). Since fundamentally quantization noise is not white but rather input signal dependent, \( e(n) \) and \( e(n-1) \), and \( e(n-1) \) and \( e(n-2) \) are correlated. \( e(n) - e(n-1) \) and \( e(n-1) - e(n-2) \) largely remove the correlated portion of the quantization noise. As a result, lower quantization noise at the output of the modulator (at frequency \( \omega \ll \omega_s \)) is obtained.
Noise of Nyquist ADCs

- **Nyquist ADCs**: Sampling frequency is twice the bandwidth of input signal.
  1. Advantage: Low power consumption due to low sampling frequency.

- **Quantization noise** is uniformly distributed over $$[-\frac{\Delta}{2}, \frac{\Delta}{2}]$$ with its power ($$P_e$$) given by

  $$P_e = \frac{\Delta^2}{12}. \quad (1)$$

- In frequency domain, since $$P_e$$ is uniformly distributed over $$[-f_B, f_B]$$ where $$f_B$$ is signal bandwidth (strictly speaking, it should be the noise bandwidth), power spectral density (PSD, $$S_{e,Nyquist}$$) of Nyquist ADCs is given by

  $$S_{e,Nyquist} = \frac{P_e}{2f_B} = \frac{\Delta^2}{12} \frac{1}{2f_B}. \quad (2)$$

Another way to look at this \(^3\).

$$P_e = \int_{-\infty}^{\infty} S_{e,Nyquist}(f) df = S_{e,Nyquist} \int_{-f_B}^{f_B} df = S_{e,Nyquist}(2f_B). \quad (3)$$

\(^3\) We assume quantization noise is white such that $$S_{e,Nyquist}$$ is constant over $$[-f_B, f_B]$. 

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Noise of Nyquist ADCs (continued)

- Power of input signal $v_{in}(t) = V_m \cos(\omega t)$

$$P_{in} = \left( \frac{V_m}{\sqrt{2}} \right)^2 = \left( \frac{FSR}{2\sqrt{2}} \right)^2 = \left( \frac{2^N \Delta}{\sqrt{2}} \right)^2.$$  \hfill (4)

- Power of quantization noise

$$P_e = \frac{\Delta^2}{12}.$$  \hfill (5)

- SNR of Nyquist ADCs

$$SNR_{Nyquist} = 10\log \left( \frac{P_{in}}{P_e} \right) = 6.02N + 1.76.$$  \hfill (6)
Noise of Oversampling ADCs

- Sampling frequency is much higher than Nyquist frequency: \( OSR = \frac{f_s}{2f_B} \).
  
  1. Pros: (i) Relaxed constraints on the stop-band attenuation of anti-aliasing filters. (ii) Improved noise performance and a better SNR.
  2. Cons: High power consumption due to high sampling frequency.

- Quantization noise power \( P_e \) is uniformly distributed over \([-f_B \times OSR, f_B \times OSR]\), power spectral density (PSD, \( S_{e,OSR} \)) of oversampling ADCs is given by

\[
S_{e,OSR} = \frac{P_e}{2f_B \times OSR} = \frac{\Delta^2}{12} \frac{1}{2f_B \times OSR} = \frac{S_{e,Nyquist}}{OSR}. \tag{7}
\]

\( \rightarrow \) Oversampling lowers PSD by factor \( OSR \).

Another way to look at this

\[
P_e = \int_{-\infty}^{\infty} S_{e,OSR}(f) df = S_{e,OSR} \int_{-f_B \times OSR}^{f_B \times OSR} df = S_{e,OSR}(2f_B \times OSR). \tag{8}
\]
Noise of Oversampling ADCs (continued)

- Power of quantization noise inside signal band $[-f_B, f_B]$

$$P_{e,\text{OSR,in-band}} = \int_{-f_B}^{f_B} S_{e,\text{OSR}}(f) df = \frac{P_e}{OSR}.$$  \hspace{1cm} (9)

→ Oversampling lowers the power of in-band quantization noise by a factor of OSR. Doubling OSR lowers the power of in-band quantization noise by 3 dB. Note the price paid for this is increased power consumption.

- SNR of oversampling ADCs

$$SNR_{OSR} = 10 \log \left( \frac{P_{in}}{P_{e,\text{OSR,in-band}}} \right)$$

$$= 10 \log \left[ \left( \frac{2^N \Delta}{2 \sqrt{2}} \right)^2 \frac{1}{\Delta^2} \frac{1}{12 \times OSR} \right]$$

$$= 6.02N + 1.76 + 10 \log(OSR).$$  \hspace{1cm} (10)

→ Doubling OSR increases SNR by 3 dB.
Noise of 1st-Order Oversampling $\Delta \Sigma$ ADCs

- Both oversampling and 1st-order noise-shaping improve noise performance.
  1. Pros: (i) Relaxed constraints on the stop-band attenuation of anti-aliasing filters. (ii) Improved noise performance (better SNR as compared with oversampling ADCs).
  2. Cons: (i) High power consumption due to high sampling frequency. (ii) Feedback operation limits the frequency of input.

Noise transfer function in signal band

$$|NTF(j\omega)| = |1 - z^{-1}|_{z = e^{i\omega T_s}} \approx \omega T_s.$$  \hspace{1cm} (11)

Power of quantization noise inside signal band

$$P_{e,\text{delta-sigma,in-band}} = \int_{-f_B}^{f_B} (\omega T_s)^2 S_{e,\text{OSR}}(f) df = \frac{\Delta^2}{12} \frac{\pi^2}{3 \times \text{OSR}^3}$$

$$= \left( \frac{\pi^2}{3 \times \text{OSR}^2} \right) P_{e,\text{OSR}}.$$  \hspace{1cm} (12)

Since $\frac{\pi^2}{3} = 3.28$, OSR needs to be greater than 1.8 in order to have $P_{e,\text{delta-sigma,in-band}} < P_{e,\text{OSR}}$. Since OSR is typically large, for example, 32 or 64, $P_{e,\text{delta-sigma,in-band}} \ll P_{e,\text{OSR}}$. 
Noise of 2nd-Order Oversampling $\Delta \Sigma$ ADCs

- Noise transfer function in signal band

$$|NTF(j\omega)| = |(1 - z^{-1})^2|_{z = e^{j\omega T_s}} \approx (\omega T_s)^2. \quad (13)$$

- Power of quantization noise inside signal band

$$P_{e, \text{delta-sigma, in-band}} = \int_{-f_B}^{f_B} (\omega T_s)^4 S_{e, \text{OSR}}(f) df = \frac{\Delta^2}{12} \frac{\pi^4}{5 \times \text{OSR}^5}$$

$$= \left( \frac{\pi^4}{5 \times \text{OSR}^4} \right) P_{e, \text{OSR}}. \quad (14)$$
Noise of Lth-Order Oversampling ΔΣ ADCs

▶ Power of quantization noise inside the signal band

\[
P_{e,\text{delta-sigma,in-band}} = \frac{\pi^{2L}}{(2L + 1) \times OSR^{2L}} P_{e,\text{OSR}}. \tag{15}
\]

▶ SNR

\[
SNR_{\text{delta-sigma}} = 10 \log \left( \frac{P_{in}}{P_{e,\text{delta-sigma,in-band}}} \right)
\]

\[
= 10 \log \left[ \left( \frac{2^N \Delta}{2\sqrt{2}} \right)^2 \frac{1}{\Delta^2 \frac{12}{(2L+1) \times OSR^{2L+1}}} \right]
\]

\[
= 6.02N + 1.76 + 10(2L + 1) \log(OSR) + 10 \log \left( \frac{2L + 1}{\pi^{2L}} \right). \tag{16}
\]

Note \(SNR_{OSR} = 6.02N + 1.76 + 10 \log(OSR)\). ΔΣ operation increases SNR.