

From the Wavelet Series to the Discrete Wavelet Transform — the Initialization

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Abstract

Discrete wavelet transform (DWT) is computed by subband filters bank and often used to approximate wavelet series (WS) and continuous wavelet transform (CWT). The approximation is often inaccurate because of the improper initialized discretization of the continuous-time signal. In this paper, the problem is analyzed and two simple algorithms for the initialization are introduced. Finally, numerical examples are presented to show that our algorithms are more effective than others.

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I. INTRODUCTION

Time-scale methods have become well known as useful tools for various signal processing applications. Continuous wavelet transform (CWT) and wavelet series (WS) are defined for continuous-time signal while discrete wavelet transform (DWT) for discrete-time signal. The CWT and WS are best suited to signal analysis and noise immunity [1], [2] because they have many good mathematical properties [3], while the DWT is usually used for signal coding applications [4], [5]. Because of the pyramidal algorithm and dyadic sampling [6], the DWT can be computed fast and efficiently, and the CWT and WS coefficients have to be approximated by the DWT in practice.

The CWT is well known as:

$$CWT_f^\psi(a, b) = \langle f, \psi^{a,b} \rangle = |a|^{-\frac{1}{2}} \int f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt \quad (1)$$

where a and b are called scale and time parameters, respectively, and

$$\psi^{a,b} = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right) \quad (2)$$

Due to the heavy redundancy, the CWT can be sampled in the time-scale plane (a, b) with the dyadic grid to form WS, i.e.,

$$\begin{aligned} WS_f^\psi(m, n) &= \langle f, \psi_{m,n} \rangle = 2^{-\frac{m}{2}} \int f(t) \overline{\psi(2^{-m}t - n)} dt \\ &= CWT_f^\psi(2^m, 2^m n) \end{aligned} \quad (3)$$

with $\psi_{m,n} = 2^{-\frac{m}{2}} \psi(2^{-m}t - n)$.

For biorthogonal (or orthogonal) wavelet bases, a multiresolution pyramidal algorithm for WS coefficients is established by Mallat [6].

$$\langle f, \tilde{\phi}_{j,k} \rangle = \sum_k \langle f, \tilde{\phi}_{j-1,k} \rangle \tilde{g}(k - 2n) \quad (4)$$

$$\langle f, \tilde{\psi}_{j,k} \rangle = \sum_k \langle f, \tilde{\phi}_{j-1,k} \rangle \tilde{h}(k - 2n) \quad (5)$$

where $\tilde{\phi}(t)$ and $\tilde{\psi}(t)$ are the dual functions of the scaling function $\phi(t)$ and wavelet $\psi(t)$, respectively, and \tilde{g} and \tilde{h} are their associated filters. $\{\tilde{\phi}_{j,k}\}_{k \in \mathbf{Z}}$ forms a basis in the multiresolution analysis subspace \mathbf{V}_j .

For a discrete-time sequence $x[n]$, $n \in \mathbf{Z}$, DWT is defined by discrete-time multiresolution decomposition [5], and can be computed by the pyramidal algorithm,

$$\nu_n^0 = x[n] \quad (6)$$

$$\nu_n^j = \sum_k \nu_k^{j-1} \tilde{g}[k - 2n], \quad j = 1, 2, \dots \quad (7)$$

$$w_n^j = \sum_k \nu_k^{j-1} \tilde{h}[k - 2n], \quad j = 1, 2, \dots \quad (8)$$

where \tilde{g} and \tilde{h} are the analysis scaling and wavelet filters, respectively.

It is shown that WS is associated with DWT by octave-band perfect reconstruction filters bank, but they are not always the same things, as pointed out by Rioul and Duhamel [7] and Shensa [8]. For coding applications, it may be appropriate to use DWT. However, the exact CWT or WS coefficients are often needed for signal analysis and data processing. Abry and Flandrin [9] have stressed the importance of performing an initialization of DWT for approximating the WS coefficients and proposed a simple algorithm to do this, but their algorithm is rough and not always effective. In this paper, we formulate this problem in Section II, and introduce two simple general algorithms for the initialization of DWT in Section III. Several simple numerical examples are given in Section IV to demonstrate the performance of our algorithms and then the results are analyzed.

II. FORMULATION OF THE INITIALIZATION PROBLEM

We have seen that DWT can be computed exactly using a computer by formula (6)-(8). We have to discretized the analog signal $f(t)$ from start when using formula (4), (5). In

fact, we compute the DWT of discretized version $f[k]$ of $f(t)$ to approximate the WS coefficients of $f(t)$. A usual way to discretize a band-limited signal $f(t)$ is to sample it, i.e.,

$$f[k] = f(t = k), k \in \mathbf{Z} \quad (9)$$

(Without loss of generality the sampling rate is assumed to be equal to one.) Then through sampling theorem, we have

$$f(t) = \sum_k f[k] \frac{\sin \pi(t - k)}{\pi(t - k)} \quad (10)$$

In more general sampling scheme, we have

$$f(t) = \sum_k f[k] \chi(t - k) \quad (11)$$

where $\chi(t)$ is a sampling function.

$f[k]$ is often used as initial value for DWT to approximate WS coefficients [6],

$$\nu_k^0 = f[k] \quad (12)$$

in this case, only if $f[k]$ is the projection of $f(t)$ onto \mathbf{V}_0 [7], [8], [9], i.e.,

$$\nu_k^0 = f[k] = \int f(t) \tilde{\phi}^*(t - k) dt \quad (13)$$

we can get exact WS coefficients of $f(t)$, otherwise, what we compute by DWT are the WS coefficients of another function $f'(t)$, where

$$f'(t) = \sum_k f[k] \phi(t - k) \quad (14)$$

In the general case, $\phi(t)$ is different from $\chi(t)$, so there may be some unexpected differences between $f'(t)$ and $f(t)$.

Rioul and Duhamel [7] use the prefilter version of $f[k]$ as the projection of $f(t)$ onto \mathbf{V}_0 ,

$$\nu_k^0 = \int f(t) \tilde{\phi}^*(t - k) dt = \sum_n f[n] \alpha[k - n] \quad (15)$$

with

$$\alpha[k] = \int \chi(t) \tilde{\phi}^*(t - k) dt \quad (16)$$

but it is not easy to calculate $\alpha[k]$ in general, and we have to look for the more appropriate algorithm.

We can consider the problem in two ways. First, we should determine which scale-limited subspace \mathbf{V}_j the signal $f(t)$ belongs to, then calculate the projection of $f(t)$ onto \mathbf{V}_j . Second, we can project $f(t)$ onto any scale-limited subspace and then get the WS coefficients. Obviously, the first way is more accurate. In fact, either Rioul and Duhamel [7] or Abry and Flandrin [9] consider the problem in the second way, i.e., they project $f(t)$ onto \mathbf{V}_0 .

Later, we present the algorithms based on the above considerations. As will be shown, our algorithms are more flexible and effective.

III. TWO ALGORITHMS OF INITIALIZATION

We use the band-limited signal as an illustration to show our algorithms. That is to assume

$$\chi(t) = \text{sinc } t = \frac{\sin \pi t}{\pi t} \quad (17)$$

To evaluate the WS coefficients of $f(t)$ exactly, we should use the projection of $f(t)$ onto some scale-limited space \mathbf{V}_J as initial value of DWT, i.e.,

$$\nu_k^0 = \langle f, \tilde{\phi}_{j,k} \rangle = \int f(t) \cdot \sqrt{2^{-J}} \tilde{\phi}^*(2^{-J}t - k) dt \quad (18)$$

Using Eq. (10), we get

$$\begin{aligned} \nu_k^0 &= \sum_n f[n] \int \text{sinc}(t - n) \cdot \sqrt{2^{-J}} \tilde{\phi}^*(2^{-J}t - k) dt \\ &= \sum_n f[n] \int \text{sinc}(t) \cdot \sqrt{2^{-J}} \tilde{\phi}^*(2^{-J}t + 2^{-J}n - k) dt \end{aligned} \quad (19)$$

Assuming

$$\alpha[m] = \langle \text{sinc}, \tilde{\phi}_{j,k} \rangle = \int \text{sinc}(t) \sqrt{2^{-J}} \tilde{\phi}^*(2^{-J}t - m) dt \quad (20)$$

we have

$$\nu_k^0 = \sum_n f[n] \alpha[k - 2^{-J}n] \quad (21)$$

To compute $\alpha[m]$, we rewrite Eq. (20) as

$$\alpha[m] = \frac{1}{2\pi} \int \Pi(\omega) \sqrt{2^J} \tilde{\Phi}^*(2^J\omega) \exp(i2^J\omega m) d\omega \quad (22)$$

where $\Pi(\omega)$ and $\tilde{\Phi}(\omega)$ are the Fourier transform of $\text{sinc}(t)$ and $\tilde{\phi}(t)$, respectively, with $\Pi(\omega) = 1$, if $|\omega| \leq \pi$ and 0 elsewhere, as indicated in Fig. 1.

For simplifying the computation, we have the following algorithms:

Algorithm 1: As $\tilde{\Phi}(\omega)$ is also low-pass filter, if $J \leq 0$ and $|J|$ is large enough, the pass-band of $\tilde{\Phi}(2^J\omega)$ is much greater than $\Pi(\omega)$, and we can roughly take $|\tilde{\Phi}(2^J\omega)| = 1$ for $|\omega| \leq \pi$. Then we have

$$\alpha[m] \approx \frac{1}{2\pi} \int \Pi(\omega) \sqrt{2^J} \exp(i2^J\omega m) d\omega = \sqrt{2^J} \text{sinc}(2^J m) \quad (23)$$

Substituting (23) to (21), we get

$$\nu_k^0 \approx \sqrt{2^J} \sum_n f[n] \text{sinc}(2^J k - n) \quad (24)$$

Because of dyadic grid, the approximation using (24) is often sufficiently accurate when $J = -1$ or -2 . In fact, our approximation is equivalent to assuming $f(t) \in \mathbf{V}_J$. The assumption is reasonable if $|J|$ is large enough because $\lim_{J \rightarrow -\infty} \mathbf{V}_J = L^2(\mathbf{R})$.

Algorithm 2: If $J \geq 0$ and $|J|$ is large enough, the pass-band of $\Pi(\omega)$ is much greater than $\tilde{\Phi}(2^J\omega)$, so we can neglect the value of $|\tilde{\Phi}(2^J\omega)|$ for $|\omega| \geq \pi$. Then we get

$$\alpha[m] \approx \frac{1}{2\pi} \int \sqrt{2^J} \tilde{\Phi}^*(2^J\omega) \exp(i2^J\omega m) d\omega = \sqrt{2^{-J}} \tilde{\phi}^*(-m) \quad (25)$$

Applying (21), we obtain

$$\nu_k^0 \approx \sqrt{2^{-J}} \sum_n f[n] \tilde{\phi}^*(2^{-J}n - k) \quad (26)$$

When $|J|$ is sufficiently large, (26) is a rather good approximation for the projection of $f(t)$ onto \mathbf{V}_J . That means that we could treat $\text{sinc}(t)$ as Dirac function in the sufficient low resolution subspace \mathbf{V}_J . In fact, the algorithm in [9] is the special case of (26) by taking $J = 0$. In that case, the approximation is often rather rough due to aliasing as we can be seen in the next section by an example.

For general sampling function $\chi(t)$, we can also use the above algorithms by changing $\text{sinc}(t)$ to $\chi(t)$. The choice of J is flexible according to actual situation and expected exactness. In the first case, the projection of $f(t)$ onto \mathbf{V}_J contains almost all information of $f(t)$ because $f(t)$ approximately belongs to \mathbf{V}_J , and so the first algorithm may be more

useful in practice. In the second case, the phase information of $\tilde{\phi}(t)$ can be maintained during the transform due to the symmetry of $\text{sinc}(t)$. However, if $\tilde{\phi}(t)$ is conjugate symmetric, i.e., the scaling filter associated with $\tilde{\phi}(t)$ is linear phase, then the phase information of the WS coefficients can also be maintained after the approximation of the first algorithm.

IV. NUMERICAL EXAMPLES

As an illustration of our algorithms, we consider a simple case. Assume $\chi(t) = \text{sinc}(t)$, and take the signal function as $f(t) = \text{sinc}(t - N)$, and then $f[n] = \delta_N$, as plotted in Fig.2. We take $\tilde{\phi}(t)$ as the scale function associated with Daubechies3 [3]. For later comparison, the coefficients of projection (i.e., the WS coefficients) of $f(t)$ onto \mathbf{V}_J , $J = -1, 0, 1$, which are computed by numerical algorithm, and their Fourier transforms are plotted in Fig. 3.

Taking $J = -1$ and using algorithm 1, we obtain the initial value ν_n^0 . Then use Eq. (7) to compute ν_n^1, ν_n^2 , which correspond to the WS coefficients of $f(t)$ in \mathbf{V}_0 and \mathbf{V}_1 ; $\nu_n^0, \nu_n^1, \nu_n^2$, and their Fourier transforms are plotted in Fig. 4. Comparing Fig. 4 with Fig. 3, we see our algorithm get a rather good approximation.

Considering $J = 1$ and applying algorithm 2, we get the initial value ν_n^0 , which correspond to the WS coefficients of $f(t)$ in \mathbf{V}_1 ; ν_n^0 and its Fourier transform is shown in Fig. 5. Comparing with Fig. 3 ($J = 1$) and Fig. 4 ($j = 2$), we see the approximation is also good.

At last, taking $J = 0$ and applying algorithm 2 (i.e., the algorithm in [9]), we obtain the initial value ν_n^0 , which correspond to the WS coefficients of $f(t)$ in \mathbf{V}_0 , as shown in Fig. 6. Comparing it with the above results (i.e., $J = 0$ in Fig. 3 and $j = 1$ in Fig. 4.), one can see the obvious aliasing occurred at high frequency (0.5Hz).

V. CONCLUSION

In this work, the initialization from the WS to DWT has been studied. We have formulated the problem and discussed the methods to solve the problem. Two algorithms for initialization have been proposed. Our algorithms are more flexible, computational efficient and provide significant improvement on the accuracy of approximation of WS coefficients over the Mallat algorithm [6] and the algorithm in [9], as shown in numerical examples. A basic prerequisite of the efficiency of our algorithms is that the sampling

scheme must maintain the information of signal as much as possible. How to choose the best multiresolution subspace for initialization is an open problem in our algorithms.

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Figure Captions List:

Fig. 1. The solid line shows the Fourier transform of the scaling function $\tilde{\phi}(t)$ and the dashed line shows the Fourier transform of the simpling function $\text{sinc}(t)$. The integral of the multiplication of two curves will be the $\alpha[m]$.

Fig. 2. The sampling function $\text{sinc}(t)$ and its discrete version $\text{sinc}(n)$, (which is often used as initial value of DWT [8]). The sampled points are shown as "+" mark.

Fig. 3. (a) The WS coefficients of $f(t)$, $J = -1, 0, 1$, which are computed by numerical algorithm (using Daubechies3 wavelet), and (b) their Fourier transforms.

Fig. 4. (a)The initial value ν_n^0 of DWT ($j = 0$), which is cumpted by algorithm 1 using $J = -1$, and the DWT ($j = 1, 2$) value ν_n^1, ν_n^2 . (b) Their Fourier transforms.

Fig. 5. (a) The initial value ν_n^0 of DWT ($j = 0$), which is cumputed by algorithm 2 using $J = 1$. (b) Its Fourier transform.

Fig. 6. (a) The initial value ν_n^0 of DWT ($j = 0$), which is cumputed by algorithm 2 using $J = 0$ (the algorithm in [9].) (b) Its Fourier transform.

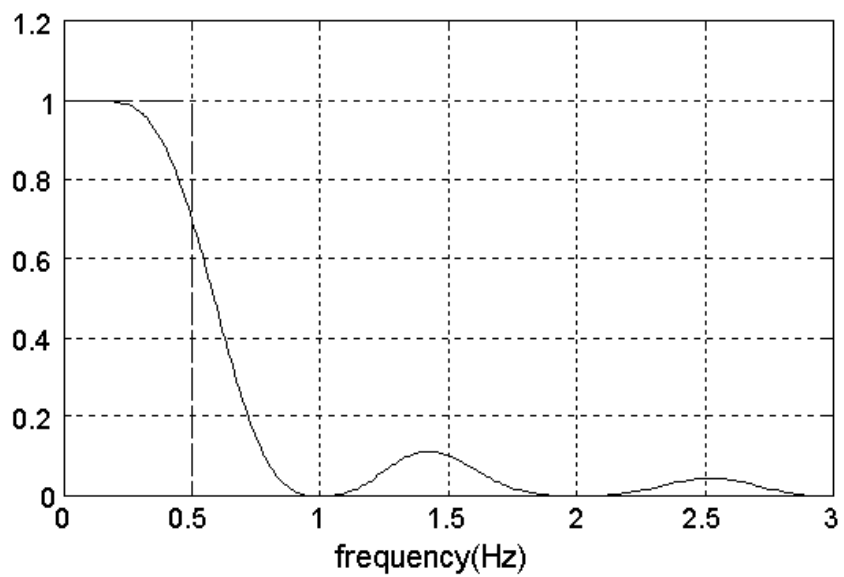


Fig. 1

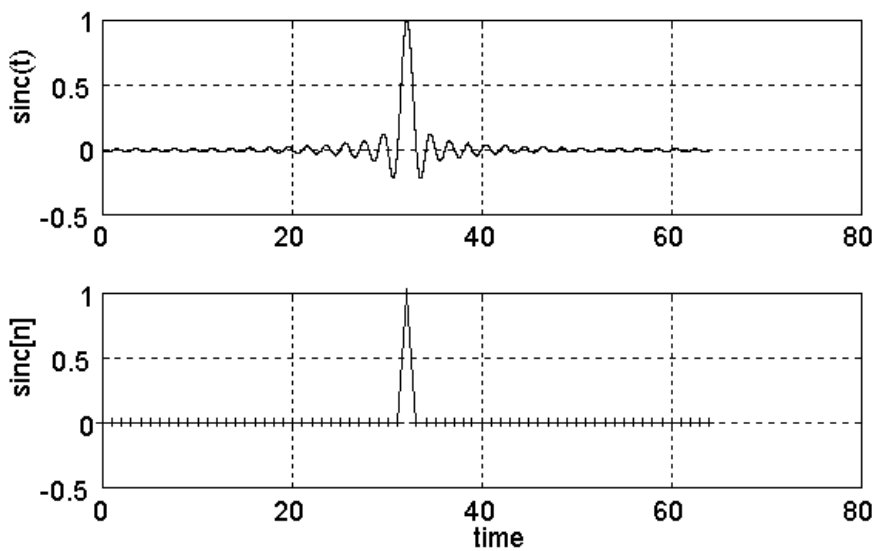


Fig. 2

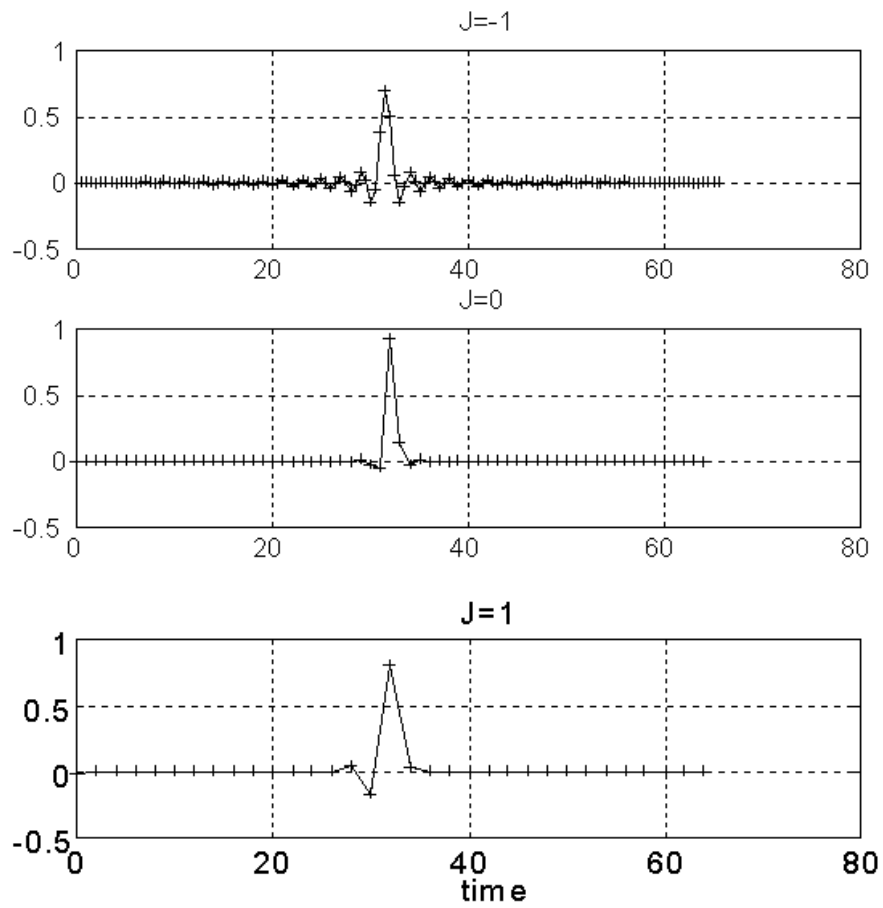


Fig. 3(a)

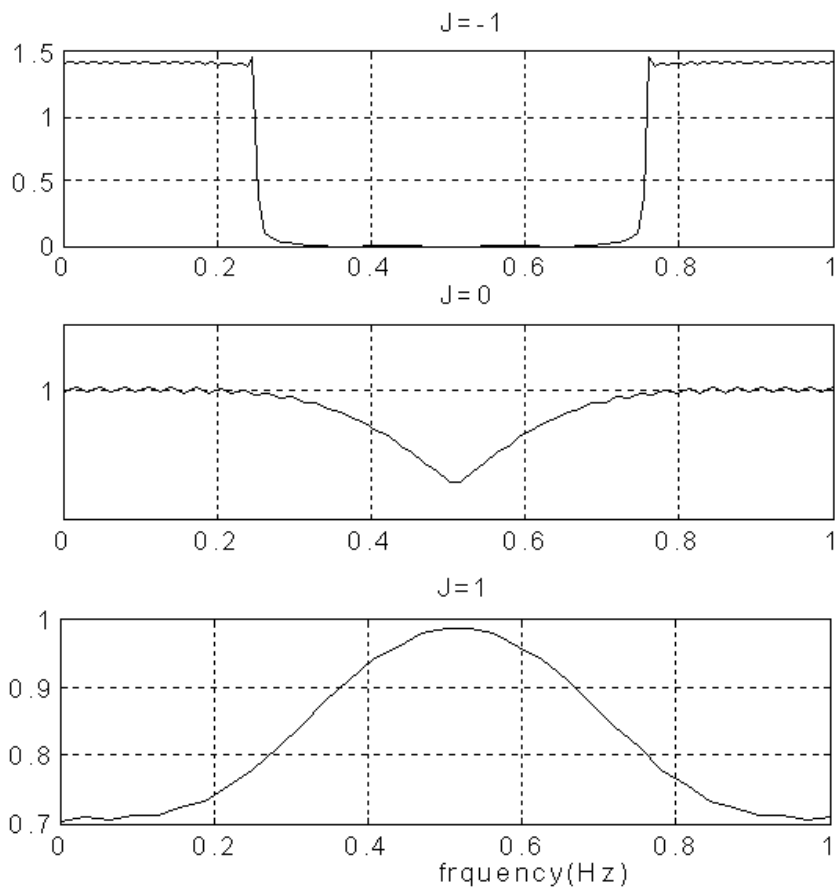


Fig. 3(b)

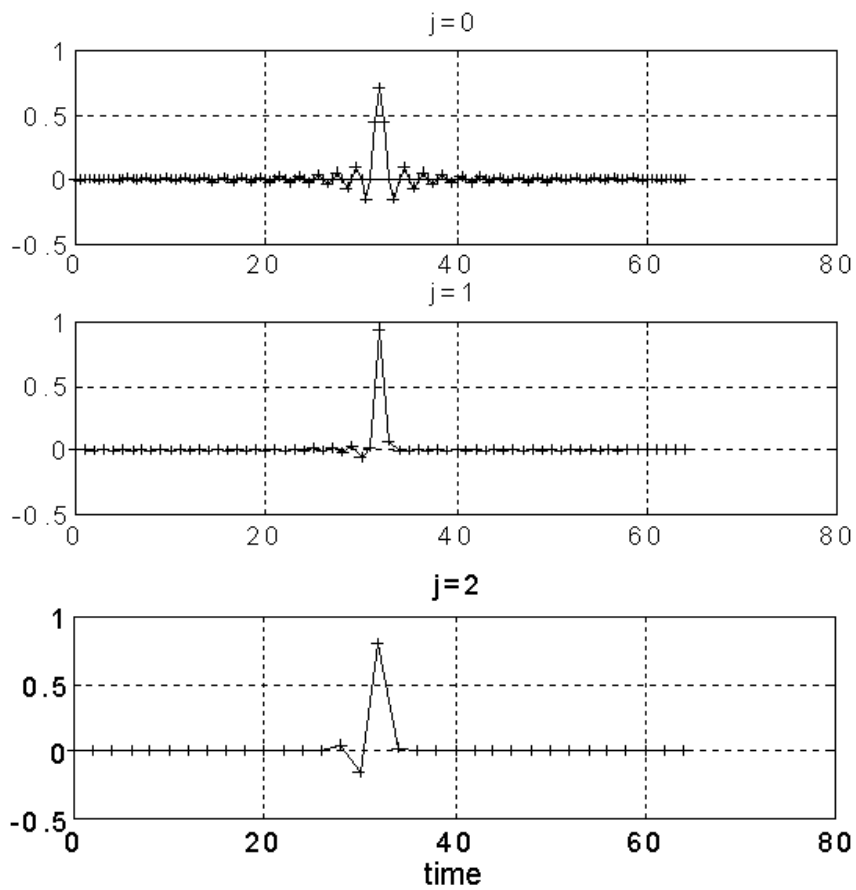


Fig. 4(a)

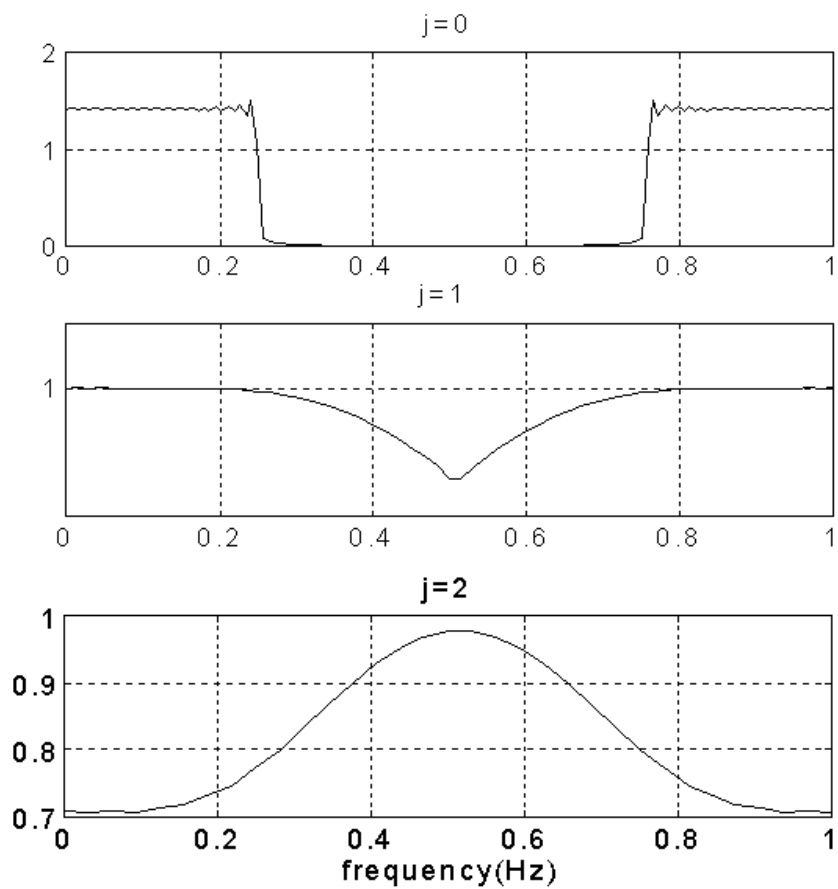


Fig. 4(b)

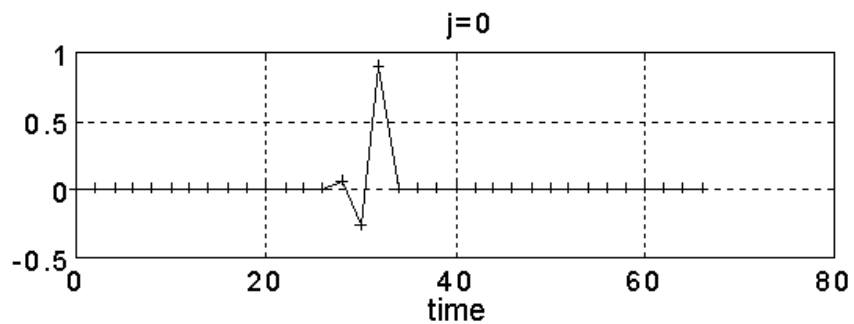


Fig. 5(a)

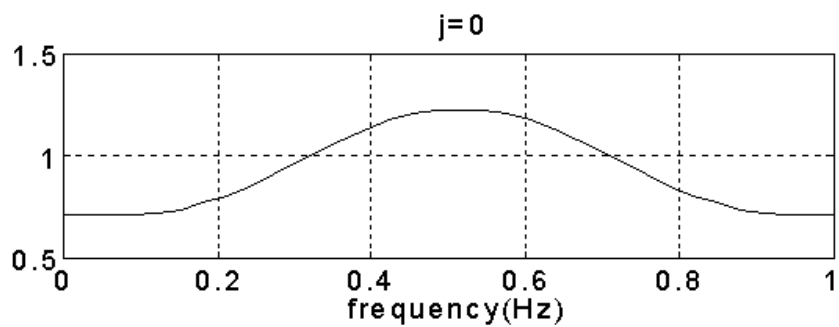


Fig. 5(b)

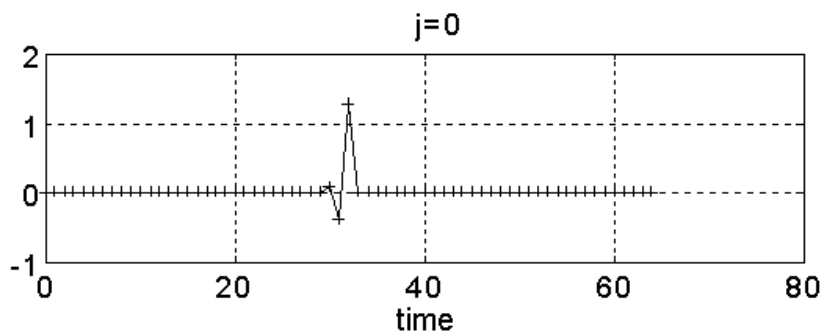


Fig. 6(a)

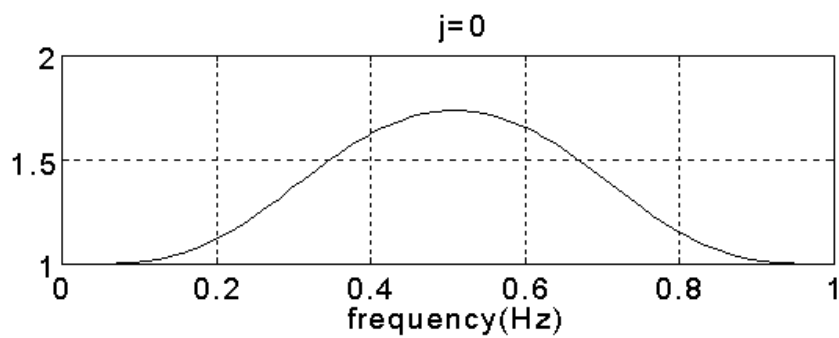


Fig. 6(b)