# Chap4 : Stochastic Processes

**Stochastic** – random **Process** – function of time

- Definition: **Stochastic Process** – A stochastic process $X(t)$ consists of an experiment with a probability measure $P[\cdot]$ defined on a sample space $S$ and a function that assigns a time function $x(t, s)$ to each outcome $s$ in the sample space of the experiment.
- Definition: **Sample Function**: A sample function $x(t, s)$ is the time function associated with outcome $s$ of an experiment.

**Example 1:**

Starting at launch time $t$=0. let $X(t)$ denote the temperature in degrees Kelvin on the surface of a space shuttle. With each launch, we record a temperature sequence $x(t,s)$. For example, $x(8073.68, 2)$=207, indicates that the temperature is 207 $K$ at 8073.68 seconds during the second launch. $X(t)$ is a stochastic process.

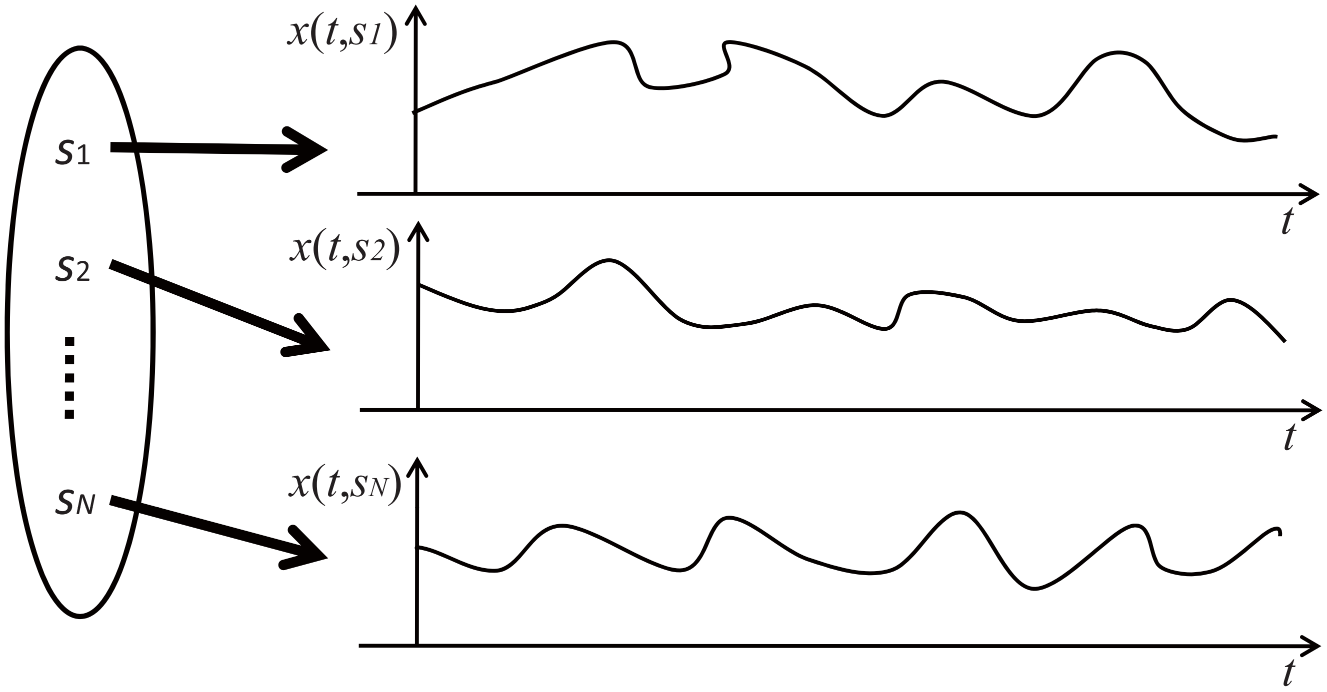


Figure 1: stochastic process representing the temperature on the surface of a space shuttle

Example 2:

Suppose that at time instants $T = 0, 1, 2, \ldots$, we roll a die and record the outcome $N_T$ where $1 \leq N_T \leq 6$. We then define the random process $X(t)$ such that for $T \leq t < T+1$, $X(t) = N_T$. In this case, the experiment consists of an infinite sequence of rolls and a sample function is just the waveform corresponding to the particular sequence of rolls. This mapping is depicted on the right.

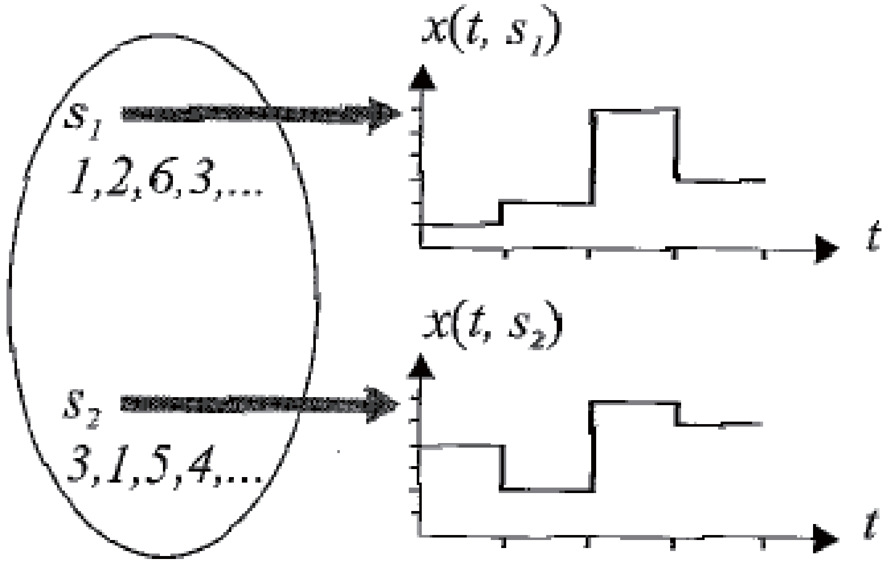


Figure 2: stochastic process representing the results of die rolls

# Types of Stochastic Processes

- **Discrete Value and Continuous Value Processes**: $X(t)$ is a discrete value process if the set of all possible values of $X(t)$ at all times $t$ is a countable set $S_X$; otherwise, $X(t)$ is a continuous value process.
- **Discrete Time and Continuous Time Process**: The stochastic process $X(t)$ is a discrete time process if $X(t)$ is defined only for a set of time instants, $t_n = nT$, where $T$ is a constant and $n$ is an integer; otherwise $X(t)$ is a continuous time process.
- Random variables from random processes: consider a sample function $x(t, s)$, each $x(t_1, s)$ is a sample value of a random variable. We use $X(t_1)$ for this random variable. The notation $X(t)$ can refer to either the random process or the random variable that corresponds to the value of the random process at time $t$.
- Example: in the experiment of repeatedly rolling a die, let $X_n = X(nT)$. What is the pmf of $X_3$?
  The random variable $X_3$ is the value of the die roll at time 3. In this case,

$$P_{X_3}(x) = \begin{cases} 1/6 & x = 1, ..., 6 \\ 0 & \text{o.w.} \end{cases}$$

Example 3: Types of Stochastic Processes

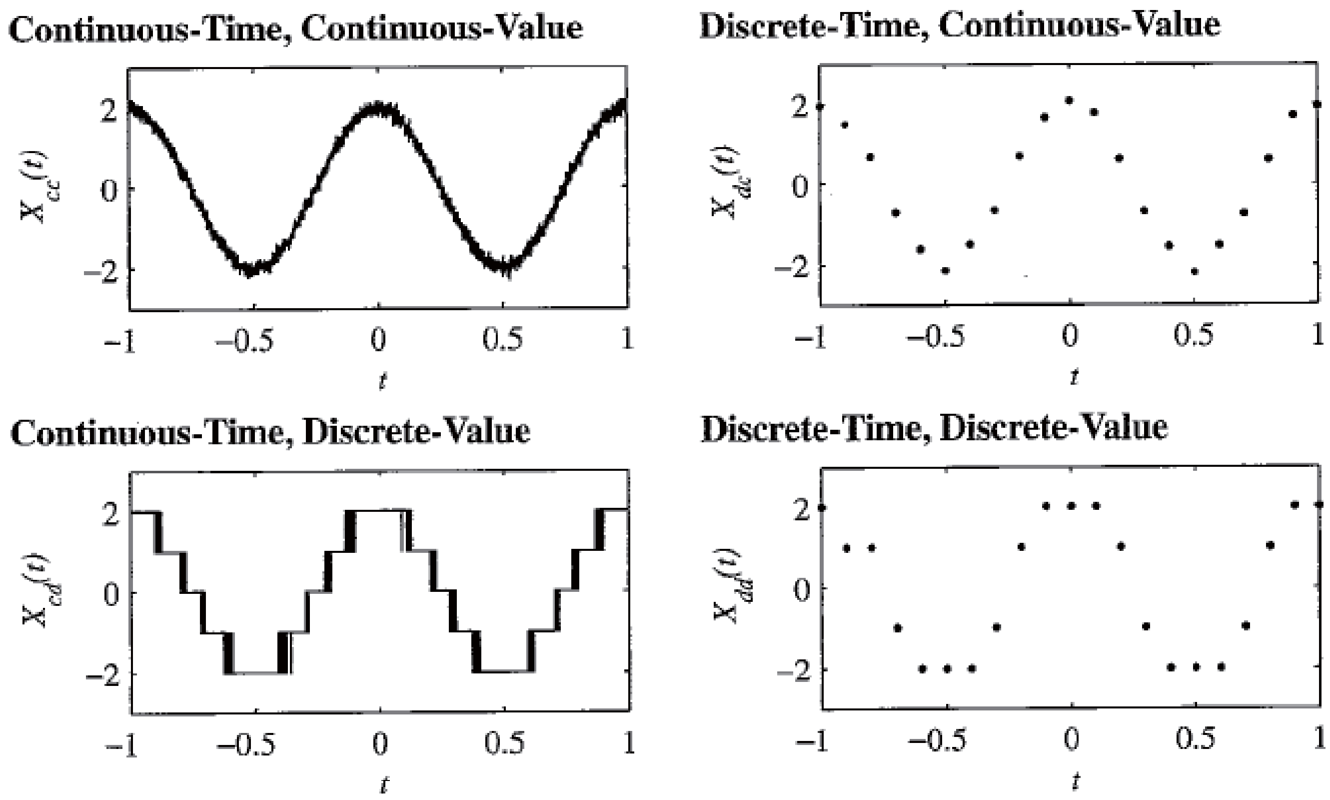


Figure 3: Sample functions of four kinds of stochastic processes. $X_{cc}(t)$ is a continuous-time, continuous-value process. $X_{dc}(t)$ is discrete-time, continuous-value process obtained by sampling $X_{cc}$) every 0.1 seconds. Rounding $X_{cc}(t)$ to the nearest integer yields $X_{cd}(t)$, a continuous-time, discrete-value process. Lastly, $X_{dd}(t)$, a discrete-time, discrete-value process, can be obtained either by sampling $X_{cd}(t)$ or by rounding $X_{dc}(t)$.

## Independent, Identically Distributed ($i.i.d$) Random Sequences

An $i.i.d.$ random sequence is a random sequence, $X_n$, in which

$$\cdots, X_{-2}, X_{-1}, X_0, X_1, X_2, \cdots$$

are $i.i.d$ random variables. An $i.i.d$ random sequence occurs whenever we perform independent trials of an experiment at a constant rate. An $i.i.d$ random sequence can be either discrete value or continuous value. In the discrete case, each random variable $X_i$ has pmf $P_{X_i}(x) = P_X(x)$, while in the continuous case, each $X_i$ has pdf $f_{X_i}(x) = f_X(x)$.

**Theorem:** Let $X_n$ denote an $i.i.d$ random sequence. For a discrete value process, the sample vector $X_{n_1}, \cdots, X_{n_k}$ has joint pmf

$$P_{X_{n_1},\cdots,X_{n_k}}(x_1, \cdots, x_k) = P_X(x_1)P_X(x_2)\cdots P_X(x_k) = \prod_{i=1}^{k} P_X(x_i)$$

Otherwise, for a continuous value process, the joint pdf of $X_{n_1}, \cdots, X_{n_k}$ is

$$f_{X_{n_1},\cdots,X_{n_k}}(x_1, \cdots, x_k) = f_X(x_1)f_X(x_2)\cdots f_X(x_k) = \prod_{i=1}^{k} f_X(x_i)$$

# $i.i.d$ Random Sequences Example

**Example 4:** For a Bernoulli process $X_n$ with success probability $p$, find the joint pmf of $X_1, \cdots, X_n$.

**Solution:** For a single sample $X_i$, we can write the Bernoulli pmf as

$$P_{X_i}(x_i) = \begin{cases} p^{x_i}(1-p)^{1-x_i} & x_i \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

When $x_i \in \{0,1\}$ for $i = 1, \cdots, n$, the joint pmf can be written as

$$P_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^k(1-p)^{n-k}$$

where $k = x_1 + \cdots + x_n$. The complete expression for the joint pmf is

$$P_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = \begin{cases} p^{x_1+\cdots+x_n}(1-p)^{n-(x_1+\cdots+x_n)} & x_i \in \{0,1\}, i = 1, 2, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

## Expected Value and Correlation

- **The Expected Value of Process:** The expected value of a stochastic process $X(t)$ is the deterministic function

$$\mu_X(t) = E[X(t)]$$

- **Autocovariance:** the autocovariance function of the stochastic process $X(t)$ is

$$C_X(t, \tau) = Cov[X(t), X(t + \tau)]$$

- **Autocorrelation:** The autocorrelation function of the stochastic process $X(t)$ is

$$R_X(t, \tau) = E[X(t)X(t + \tau)]$$

- Autocovariance and Autocorrelation of a Random Sequence:

$$C_X[m, k] = Cov[X_m, X_{m+k}] = R_X[m, k] - E[X_m]E[X_{m+k}]$$

where $m$ and $k$ are integers. the autocorrelation function of the random sequence $X_n$ is

$$R_X[m, k] = E[X_m X_{m+k}]$$

**Example 5:** The input to a digital filter is an $i.i.d$ random sequence $\cdots, X_{-1}, X_0, X_1, \cdots$ with $E[X_i] = 0$ and Var[$X_i$]=1. The output is also a random sequence $\cdots, Y_{-1}, Y_0, Y_1, \cdots$. The relationship between the input sequence and output sequence is expressed in the formula

$$Y_n = X_n + X_{n-1} \qquad \text{for all integer } n$$

Find the expected value function $E[Y_n]$ and autocovariance function $C_Y(m, k)$ of the output.

**Solution:** Because $Y_i = X_i + X_{i-1}$, we have $E[Y_i] = E[X_i] + E[X_{i-1}] = 0$. Before calculating $C_Y[m, k]$, we observe that $X_n$ being an i.i.d random sequence with $E[X_n] = 0$ and Var[$X_n$]=1 implies

$$C_X[m, k] = E[X_m X_{m+k}] = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

For any integer $k$, we can write

$$\begin{aligned}
C_Y[m,k] &= E[Y_m Y_{m+k}] = E[(X_m + X_{m-1})(X_{m+k} + X_{m+k-1})] \\
&= E[X_m X_{m+k} + X_m X_{m+k-1} + X_{m-1} X_{m+k} + X_{m-1} X_{m+k-1}] \\
&= E[X_m X_{m+k}] + E[X_m X_{m+k-1}] + E[X_{m-1} X_{m+k}] + E[X_{m-1} X_{m+k-1}] \\
&= C_X[m,k] + C_X[m,k-1] + C_X[m-1,k+1] + C_X[m-1,k]
\end{aligned}$$

We still need to evaluate the above expression for all $k$. For each value of $k$, some terms in the above expression will equal zero since $C_X[m,k] = 0$ for $k \neq 0$.

- When $k = 0$,

$$C_Y[m,0] = C_X[m,0] + C_X[m,-1] + C_X[m-1,1] + C_X[m-1,0] = 2.$$

- When $k = 1$

$$C_Y[m,1] = C_X[m,1] + C_X[m,0] + C_X[m-1,2] + C_X[m-1,1] = 1.$$

- When $k = -1$

$$C_Y[m,-1] = C_X[m,-1] + C_X[m,-2] + C_X[m-1,0] + C_X[m-1,-1] = 1.$$

- When $k = 2$

$$C_Y[m,2] = C_X[m,2] + C_X[m,1] + C_X[m-1,3] + C_X[m-1,2] = 0.$$

A complete expression for the autocovariance is

$$C_Y[m,k] = \begin{cases} 2 & k = 0 \\ 1 & k = \pm 1 \\ 0, \text{otherwise} & \end{cases}$$