Assignment 2

1. We place at random \( n \) points in the interval \((0,1)\) and we denote by random variables \( X \) and \( Y \) the distance from the origin to the first and the last points respectively. Find \( F_X(x) \), \( F_Y(y) \) and \( F_{X,Y}(x,y) \).

2. A random variable \( X \) has the density function

\[
f_X(x) = \begin{cases} \frac{1}{2} \exp \left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{o.w.} \end{cases}
\]

Define events \( A = \{1 < X \leq 3\} \), \( B = \{X \leq 2.5\} \), and \( C = A \cap B \). Find the probabilities of events \( A \), \( B \), and \( C \).

3. Suppose height to the bottom of clouds is a Gaussian R.V. \( X \) for which \( \mu = 4000 \text{m} \), and \( \sigma = 1000 \text{m} \). A person bets that cloud height tomorrow will fall in the set \( A = \{1000 \text{m} < X \leq 3300 \text{m}\} \) while a second person bets that height will be satisfied by \( B = \{2000 \text{m} < X \leq 4200 \text{m}\} \). A third person bets they are both correct. Find the probabilities that each person will win the bet.

4. A random variable \( X \) is known to be Poisson with \( \lambda = 4 \).

   (a) Plot the density and distribution functions for this random variable.

   (b) What is the probability of the events \( \{0 \leq X \leq 5\} \)?

5. A random variable \( X \) has a probability density

\[
f_X(x) = \begin{cases} \frac{\pi}{16} \cos \left(\frac{\pi x}{8}\right) & -4 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}
\]

Find: (a) its mean value \( \bar{X} \), (b) its second moment \( \bar{X^2} \), and (c) its variance.

6. A random variable has a probability density

\[
f_X(x) = \begin{cases} \frac{5}{4} (1 - x^4) & 0 < x \leq 1 \\ 0 & \text{o.w.} \end{cases}
\]

Find: (a) \( E[X] \), (b) \( E[4X^2] \), and (c) \( E[X^2] \).
7. Suppose a coin having probability 0.7 of coming up heads is tossed three times. Let $X$ denote the number of heads that appear in the three tosses. Determine the probability mass function of $X$.

8. If the distribution function of $F$ is given by

$$F(b) = \begin{cases} 
0, & b < 0 \\
1/2, & 0 \leq b < 1 \\
3/5, & 1 \leq b < 2 \\
4/5, & 2 \leq b < 3 \\
9/10, & 3 \leq b < 3.5 \\
1, & b \geq 3.5 
\end{cases} \tag{4}$$

calculate and sketch the probability mass function of $X$.

9. On a multiple-choice exam with three possible answers for each of the five questions, what is the probability that a student would get four or more correct answers just by guessing?

10. Let $X$ be a Poisson random variable with parameter $\lambda$. Show that $P(X = i)$ increases monotonically and then decreases monotonically as $i$ increases, reaching its maximum when $i$ is the largest integer not exceeding $\lambda$. \textbf{Hints}: consider $P(X = i)/P(X = i - 1)$.

11. Let $c$ be a constant. Show that

(a) $\text{Var}(cX) = c^2 \text{Var}(X)$.

(b) $\text{Var}(c + X) = \text{Var}(X)$.

12. Suppose that $X$ takes on each of the values 1, 2, and 3 with probability 1/3. What is the moment generating function? Derive $E[X]$, $E[X^2]$, and $E[X^3]$ by differentiating the moment generating function and then compare the obtained result with a direct derivation of these moments.