Stochastic Process – function of time

- Definition: Stochastic Process – A stochastic process $X(t)$ consists of an experiment with a probability measure $P[.]$ defined on a sample space $S$ and a function that assigns a time function $x(t, s)$ to each outcome $s$ in the sample space of the experiment.

- Definition: Sample Function: A sample function $x(t, s)$ is the time function associated with outcome $s$ of an experiment.
Example 1:

Starting at launch time \( t=0 \), let \( X(t) \) denote the temperature in degrees Kelvin on the surface of a space shuttle. With each launch, we record a temperature sequence \( x(t,s) \). For example, \( x(8073.68, 2)=207 \), indicates that the temperature is 207 K at 8073.68 seconds during the second launch. \( X(t) \) is a stochastic process.

Figure 1: stochastic process representing the temperature on the surface of a space shuttle
Example 2:

Suppose that at time instants $T = 0, 1, 2, \ldots$, we roll a die and record the outcome $N_T$ where $1 \leq N_T \leq 6$. We then define the random process $X(t)$ such that for $T \leq t < T + 1$, $X(t) = N_T$. In this case, the experiment consists of an infinite sequence of rolls and a sample function is just the waveform corresponding to the particular sequence of rolls. This mapping is depicted on the right.

Figure 2: stochastic process representing the results of die rolls
Types of Stochastic Processes

- **Discrete Value and Continuous Value Processes**: $X(t)$ is a discrete value process if the set of all possible values of $X(t)$ at all times $t$ is a countable set $S_X$; otherwise, $X(t)$ is a continuous value process.

- **Discrete Time and Continuous Time Process**: The stochastic process $X(t)$ is a discrete time process if $X(t)$ is defined only for a set of time instants, $t_n = nT$, where $T$ is a constant and $n$ is an integer; otherwise $X(t)$ is a continuous time process.

- Random variables from random processes: consider a sample function $x(t, s)$, each $x(t_1, s)$ is a sample value of a random variable. We use $X(t_1)$ for this random variable. The notation $X(t)$ can refer to either the random process or the random variable that corresponds to the value of the random process at time $t$.

- Example: in the experiment of repeatedly rolling a die, let $X_n = X(nT)$. What is the pmf of $X_3$? The random variable $X_3$ is the value of the die roll at time 3. In this case,

$$P_{X_3}(x) = \begin{cases} 
1/6 & x = 1, \ldots, 6 \\
0 & \text{o.w.}
\end{cases}$$
Example 3: Types of Stochastic Processes

Figure 3: Sample functions of four kinds of stochastic processes. $X_{CC}(t)$ is a continuous-time, continuous-value process. $X_{dc}(t)$ is discrete-time, continuous-value process obtained by sampling $X_{cc}$ every 0.1 seconds. Rounding $X_{cc}(t)$ to the nearest integer yields $X_{cd}(t)$, a continuous-time, discrete-value process. Lastly, $X_{dd}(t)$, a discrete-time, discrete-value process, can be obtained either by sampling $X_{cd}(t)$ or by rounding $X_{dc}(t)$.
Independent, Identically Distributed (i.i.d) Random Sequences

An *i.i.d.* random sequence is a random sequence, $X_n$, in which

$$
\cdots, X_{-2}, X_{-1}, X_0, X_1, X_2, \cdots
$$

are *i.i.d* random variables. An *i.i.d* random sequence occurs whenever we perform independent trials of an experiment at a constant rate. An *i.i.d* random sequence can be either discrete value or continuous value. In the discrete case, each random variable $X_i$ has pmf

$$
P_{X_i}(x) = P_{X}(x),
$$

while in the continuous case, each $X_i$ has pdf

$$
f_{X_i}(x) = f_{X}(x).
$$

**Theorem:** Let $X_n$ denote an *i.i.d* random sequence. For a discrete value process, the sample vector $X_{n_1}, \cdots, X_{n_k}$ has joint pmf

$$
P_{X_{n_1}, \cdots, X_{n_k}}(x_1, \cdots, x_k) = P_X(x_1)P_X(x_2) \cdots P_X(x_k) = \prod_{i=1}^{k} P_X(x_i)
$$

Otherwise, for a continuous value process, the joint pdf of $X_{n_1}, \cdots, X_{n_k}$ is

$$
f_{X_{n_1}, \cdots, X_{n_k}}(x_1, \cdots, x_k) = f_X(x_1)f_X(x_2) \cdots f_X(x_k) = \prod_{i=1}^{k} f_X(x_i)$$
Example 4: For a Bernoulli process $X_n$ with success probability $p$, find the joint pmf of $X_1, \cdots, X_n$.

**Solution:** For a single sample $X_i$, we can write the Bernoulli pmf as

$$P_{X_i}(x_i) = \begin{cases} 
  p^{x_i} (1 - p)^{1-x_i} & x_i \in \{0, 1\} \\
  0 & \text{otherwise}
\end{cases}$$

When $x_i \in \{0, 1\}$ for $i = 1, \cdots, n$, the joint pmf can be written as

$$P_{X_1, \cdots, X_n}(x_1, \cdots, x_n) = \prod_{i=1}^{n} p^{x_i} (1 - p)^{1-x_i} = p^k (1 - p)^{n-k}$$

where $k = x_1 + \cdots + x_n$. The complete expression for the joint pmf is

$$P_{X_1, \cdots, X_n}(x_1, \cdots, x_n) = \begin{cases} 
  p^{x_1+\cdots+x_n} (1 - p)^{n-(x_1+\cdots+x_n)} & x_i \in \{0, 1\}, i = 1, 2, \cdots, n \\
  0 & \text{otherwise}
\end{cases}$$
Expected Value and Correlation

- **The Expected Value of Process:** The expected value of a stochastic process $X(t)$ is the deterministic function

\[ \mu_X(t) = E[X(t)] \]

- **Autocovariance:** the autocovariance function of the stochastic process $X(t)$ is

\[ C_X(t, \tau) = Cov[X(t), X(t + \tau)] \]

- **Autocorrelation:** The autocorrelation function of the stochastic process $X(t)$ is

\[ R_X(t, \tau) = E[X(t)X(t + \tau)] \]

- **Autocovariance and Autocorrelation of a Random Sequence:**

\[ C_X[m, k] = Cov[X_m, X_{m+k}] = R_X[m, k] - E[X_m]E[X_{m+k}] \]

where $m$ and $k$ are integers. the autocorrelation function of the random sequence $X_n$ is

\[ R_X[m, k] = E[X_mX_{m+k}] \]
Example 3: The input to a digital filter is an i.i.d random sequence $\cdots, X_{-1}, X_0, X_1, \cdots$ with $E[X_i] = 0$ and $\text{Var}[X_i] = 1$. The output is also a random sequence $\cdots, Y_{-1}, Y_0, Y_1, \cdots$. The relationship between the input sequence and output sequence is expressed in the formula

$$Y_n = X_n + X_{n-1}$$

for all integer $n$.

Find the expected value function $E[Y_n]$ and autocovariance function $C_Y(m, k)$ of the output.

Solution: Because $Y_i = X_i + X_{i-1}$, we have $E[Y_i] = E[X_i] + E[X_{i-1}] = 0$. Before calculating $C_Y(m, k)$, we observe that $X_n$ being an i.i.d random sequence with $E[X_n] = 0$ and $\text{Var}[X_n] = 1$ implies

$$C_X[m, k] = E[X_m X_{m+k}] = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$
For any integer $k$, we can write

\[
C_Y[m, k] = E[Y_mY_{m+k}] = E[(X_m + X_{m-1})(X_{m+k} + X_{m+k-1})]
\]

\[
= E[X_mX_{m+k} + X_mX_{m+k-1} + X_{m-1}X_{m+k} + X_{m-1}X_{m+k-1}]
\]

\[
= E[X_mX_{m+k}] + E[X_mX_{m+k-1}] + E[X_{m-1}X_{m+k}] + E[X_{m-1}X_{m+k-1}]
\]

\[
= C_X[m, k] + C_X[m, k-1] + C_X[m-1, k+1] + C_X[m-1, k]
\]

We still need to evaluate the above expression for all $k$. For each value of $k$, some terms in the above expression will equal zero since $C_X[m, k] = 0$ for $k \neq 0$.

- When $k = 0$, 

  
  \[
  C_Y[m, 0] = C_X[m, 0] + C_X[m, -1] + C_X[m-1, 1] + C_X[m-1, 0] = 2.
  \]

- When $k = 1$

  \[
  C_Y[m, 1] = C_X[m, 1] + C_X[m, 0] + C_X[m-1, 2] + C_X[m-1, 1] = 1.
  \]

- When $k = -1$

  \[
  C_Y[m, -1] = C_X[m, -1] + C_X[m, -2] + C_X[m-1, 0] + C_X[m-1, -1] = 1.
  \]
• When $k = 2$

$$C_Y[m, 2] = C_X[m, 2] + C_X[m, 1] + C_X[m - 1, 3] + C_X[m - 1, 2] = 0.$$ 

A complete expression for the autocovariance is

$$C_Y[m, k] = \begin{cases} 
2 & k = 0 \\
1 & k = \pm 1 \\
0, \text{ otherwise} 
\end{cases}$$