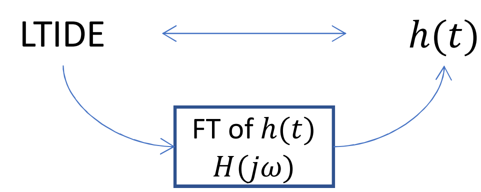
Signals and Systems I

Topic 11

Laplace Transform

Fourier Transform and LTIDE



Reminder:

What is the output of the following system to $x(t)$?

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} - 6x(t)$$

First you need to find $h(t)$ that is the "impulse response" of this system and the output is $y(t) = x(t) * h(t)$.

Alternative approach using the FT is to first find $H(j\omega)$ from the equation and then calculate inverse FT of $H(j\omega)X(j\omega)$ which is $y(t)$.

This method can be used for cases where M (order of highest derivate of input) is even greater than N (system order that is the order of highest derivative of output):

What is $h(t)$ impulse response of this system?

$$\frac{dy(t)}{dt} + 4y(t) = \frac{d^2x(t)}{dt} + \frac{dx(t)}{dt} - 6x(t)$$

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Laplace Transform

Fourier Transform provides a linear combination of $e^{j\omega t}$s for a signal:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Laplace Transform is an extension of this method and finds correlation of e^{st} where $s = \sigma + j\omega$ (in FT $s = j\omega$ only), with the signal.

$$X(j\omega) = \underbrace{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}_{s=j\omega} \xrightarrow{\text{extension to Laplace}} X(s) = \underbrace{\int_{-\infty}^{\infty} x(t)e^{-st}dt}_{s=j\omega+\sigma}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \xrightarrow{\text{extension to Laplace}} x(t) = \underbrace{\frac{1}{2\pi j}\int_{c-\infty}^{c+\infty} X(s)e^{st}ds}_{s=c+j\omega,\ c\ is\ a\ constant}$$

set c to zero and its FT

Similar to FT, Laplace is a linear operation.

$$x_1(t) \rightarrow X_1(s)$$
$$x_2(t) \rightarrow X_2(s)$$
$$ax_1(t) + bx_2(t) \rightarrow aX_1(s) + bX_2(s)$$

Why Laplace?

Many signals don't have FT but have Laplace Transform!

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Laplace Transform

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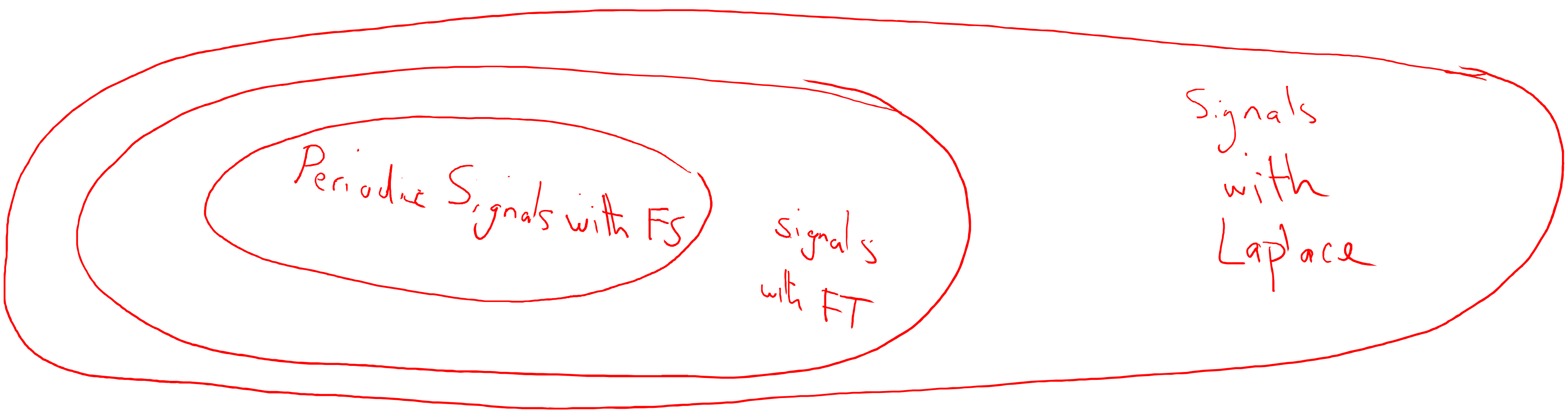

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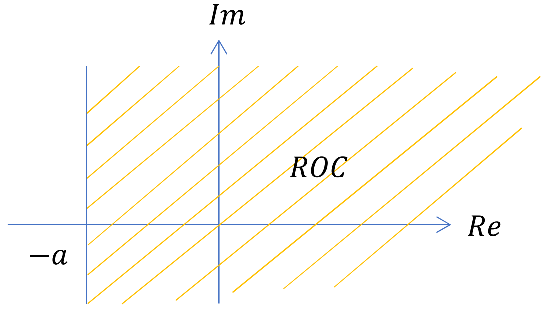
Region of Convergence (ROC) of Laplace Transform

Set of s values in complex plane that makes the integral of $X(s)$ converge.

Example: $x(t) = e^{-at}u(t),\ a > 0$. Find $X(s)$ & its ROC.

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(a+s)t}dt$$
$$= \frac{e^{-(a+s)t}}{(a+s)}\Big|_0^{\infty} = \frac{e^{-(a+s)\infty}}{-(a+s)} + \frac{1}{a+s}$$



$$\lim_{t\to\infty} \frac{e^{-(a+s)t}}{-(a+s)} = \lim_{t\to\infty} \frac{e^{-(a+\sigma+j\omega)t}}{-(a+s)}$$
$$= \lim_{t\to\infty} \frac{e^{-j\omega t}e^{-(a+\sigma)t}}{-(a+s)}$$

$e^{-j\omega t}$ is a complex number, $|e^{-j\omega t}| = 1$ for all t.

$$= \lim_{t\to\infty} \frac{e^{-(a+\sigma)t}}{-(a+s)} = \begin{cases} 0 & a+\sigma > 0 \quad \text{converges (ROC)} \\ \infty & a+\sigma < 0 \quad \text{does not converge} \end{cases}$$

$$\Rightarrow \text{ROC} = \{\text{"}s = \sigma + j\omega\text{" such that } \sigma + a > 0 \Rightarrow \sigma > -a\}$$

$x(t) = e^{-at}u(t), \quad a > 0 \quad \xrightarrow{L} \quad \frac{1}{s+a}$

ROC$= \{s, Re\{s\} > -a, \quad a > 0\}$

If $s = j\omega$ is in ROC then FT exists

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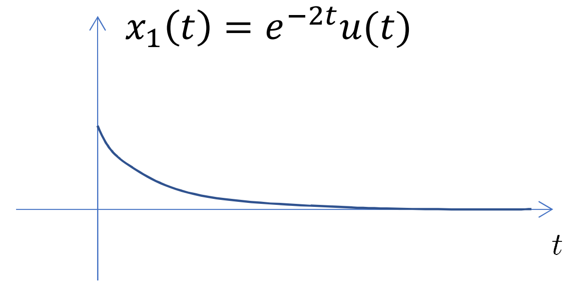
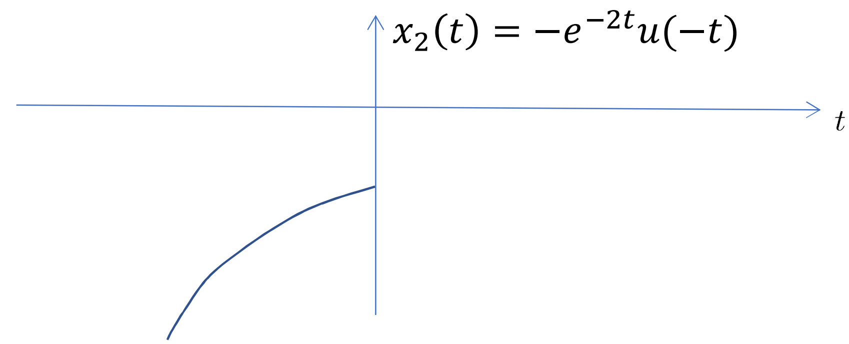
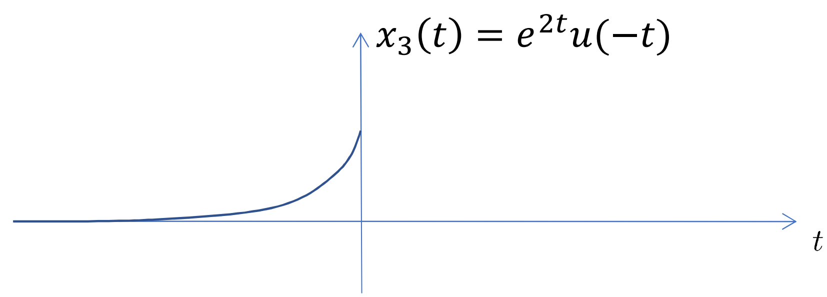
Laplace Transform & ROC

Example:
Find Laplace and ROC for the following signals:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$







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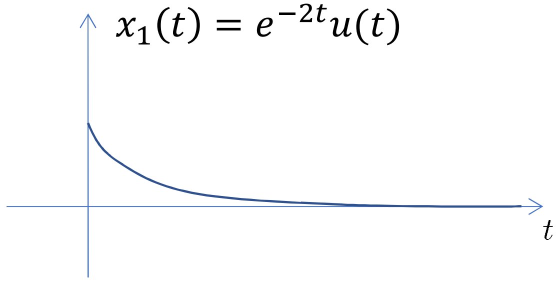
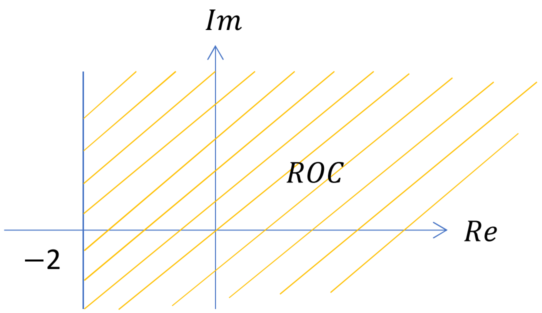
Laplace Transform & ROC



$$X_1(s) = \int_0^\infty e^{-(2+s)t} dt = \frac{1}{2+s} + \underbrace{\frac{e^{-(2+s)\infty}}{-(2+s)}}_{0 \text{ iff } 2+\sigma>0}$$

$$\text{ROC}=\{s|\ Re\{s\} > -2\} \to X_1(s) = \frac{1}{2+s}$$



*Note: Direction of the dashed area of ROC is the same as direction of the signal in time.

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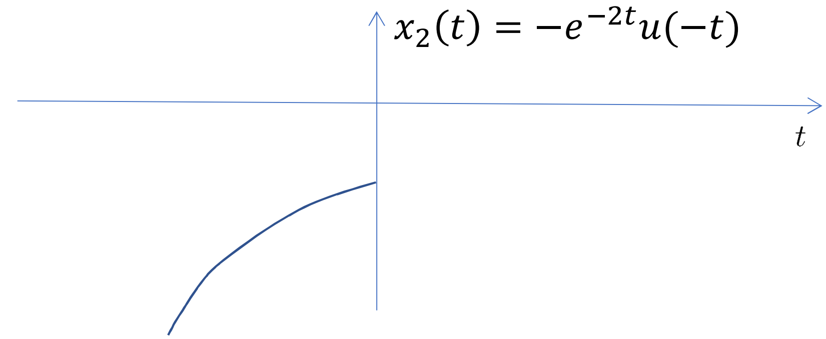
Laplace Transform & ROC

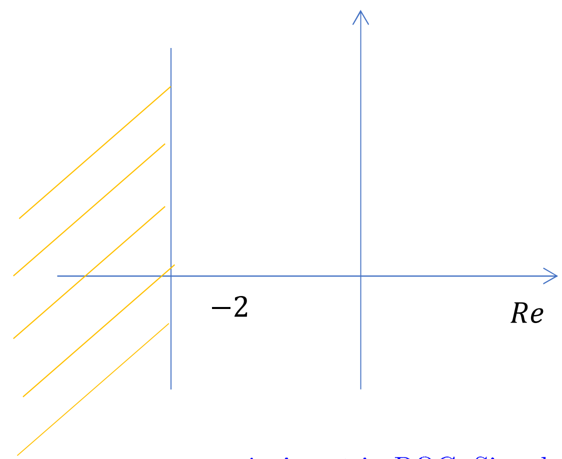
$$X_2(s) = \int_{-\infty}^{0} -e^{-2t}e^{-st}dt = -\int_{-\infty}^{0} e^{-(2+s)t}dt$$

$$= \frac{-e^{-(2+s)t}}{-(2+s)}\Big|_{-\infty}^{0} = \frac{1}{(2+s)} + \frac{e^{-(2+s)(-\infty)}}{-(2+s)}$$

$$e^{(2+s)(\infty)} = \begin{cases} 0 & 2+\sigma < 0 \rightarrow \text{ROC} = Re\{s\} < -2 \\ \infty & 2+\sigma > 0 \end{cases}$$

$$X_2(s) = \frac{1}{2+s}, \quad Re\{s\} < -2$$





Laplace is the same as that of $x_1(t)$, only ROC is different!

$s = j\omega$ is not in ROC. Signal doesn't have FT.

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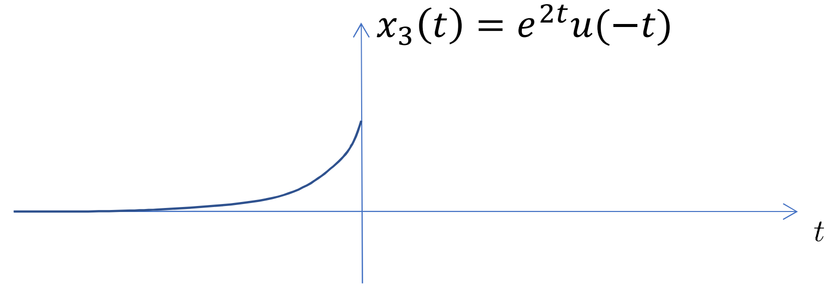
Laplace Transform & ROC

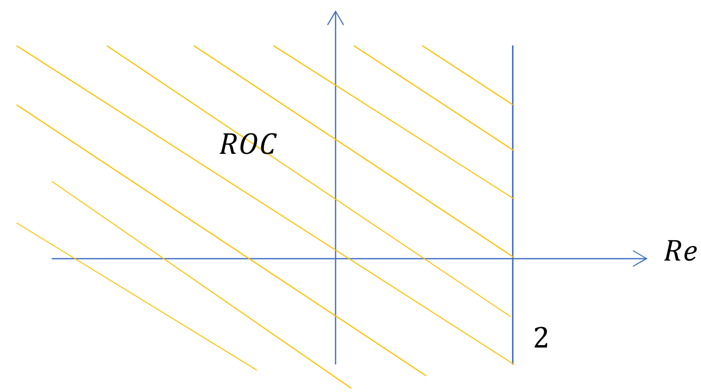
$$X_3(s) = \int_{-\infty}^{0} e^{2t} e^{-st} dt = \int_{-\infty}^{0} e^{(2-s)t} dt$$

$$= \frac{e^{(2-s)t}}{(2-s)}\Bigg|_{-\infty}^{0} = \frac{1}{(2-s)} - \frac{e^{-(2-s)(\infty)}}{(2-s)}$$

$$e^{-(2-s)(\infty)} = \begin{cases} 0 & 2-\sigma > 0 \rightarrow \text{ROC} = Re\{s\} < 2 \\ \infty & 2-\sigma < 0 \end{cases}$$

$$X_3(s) = \frac{1}{2-s}, \quad Re\{s\} < 2$$





$s = j\omega$ is in ROC. This noncausal signal has FT.

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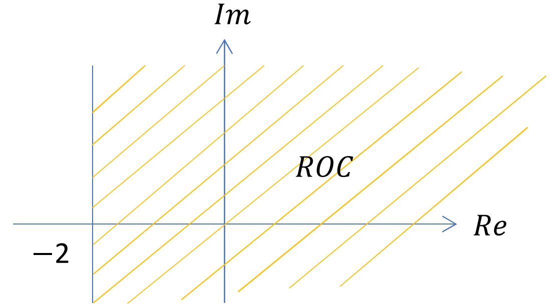
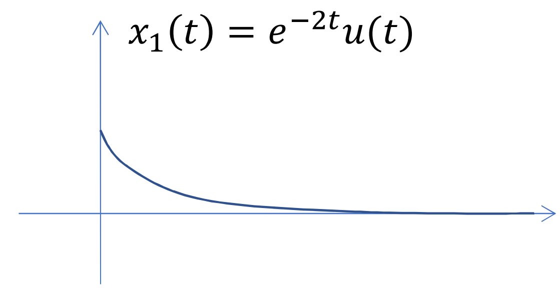
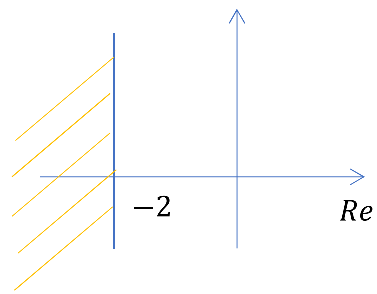
Laplace Transform & ROC

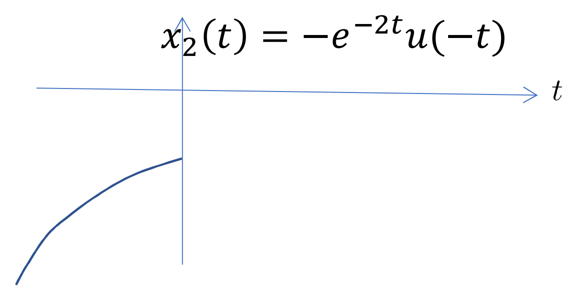
$$X_2(s) = \frac{1}{2+s}, \quad Re\{s\} > -2$$



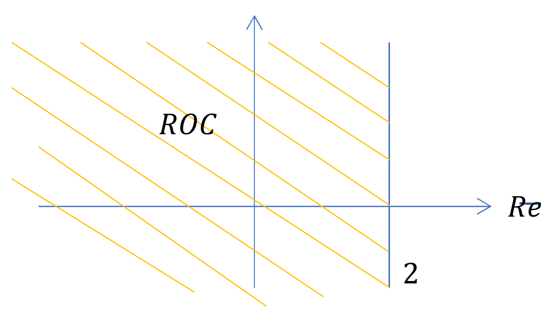


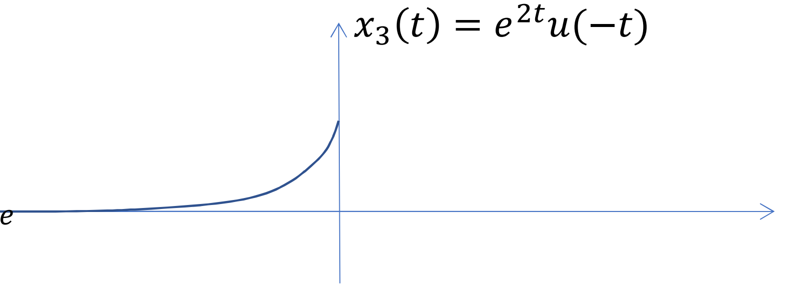
$$X_2(s) = \frac{1}{2+s}, \quad Re\{s\} < -2$$





$$X_3(s) = \frac{1}{2-s}, \quad Re\{s\} < 2$$





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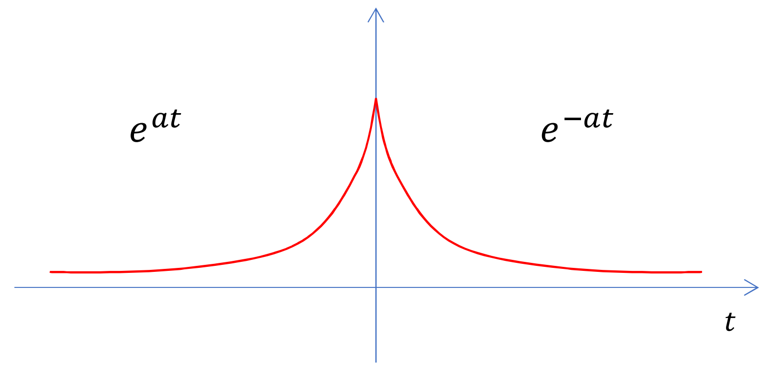
Laplace Transform & ROC

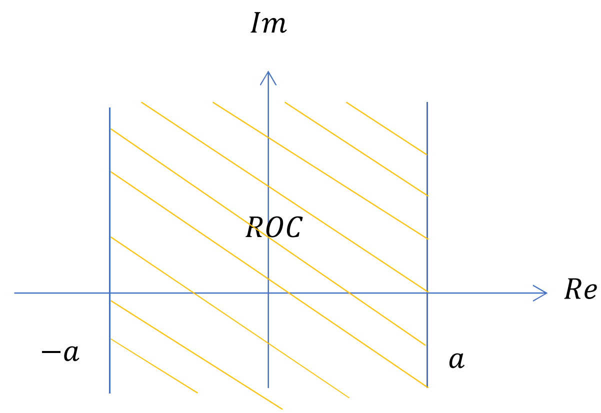
Example:
Find $X(s)$ for $e^{-a|t|}$, $a > 0$.



$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$
$$X(s) = \mathscr{L}\left(e^{-at}u(t)\right) + \mathscr{L}\left(e^{at}u(-t)\right)$$

$$ROC\left(X(s)\right) = ROC\{\mathscr{L}\left(e^{-at}u(t)\right)\} \cap ROC\{\mathscr{L}\left(e^{at}u(-t)\right)\}$$

$$e^{-at}u(t) \xrightarrow{\mathscr{L}} \frac{1}{a+s},\ Re\{s\} > -a$$
$$e^{at}u(-t) \xrightarrow{\mathscr{L}} \frac{1}{a-s},\ Re\{s\} < a$$



$$X(s) = \frac{1}{a+s} + \frac{1}{a-s} = \frac{2a}{a^2 - s^2}$$

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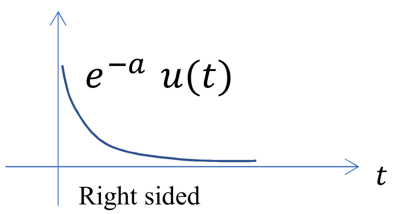
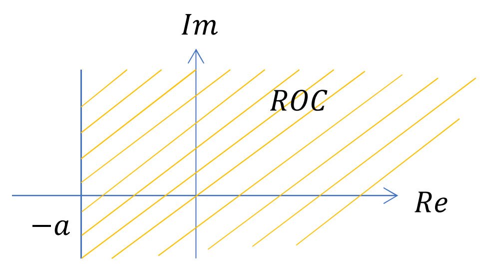
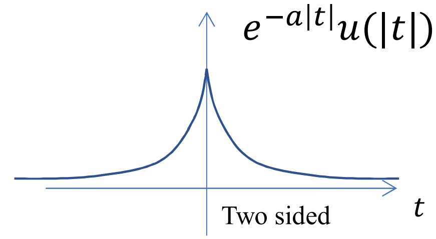
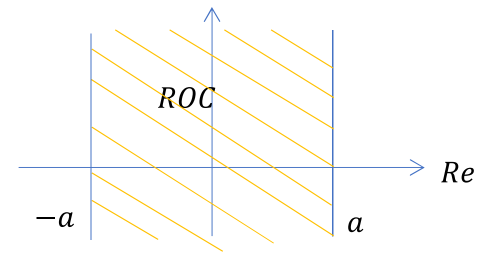
Laplace Transform & ROC

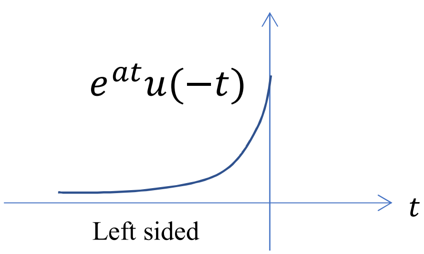
Left sided, Right sided or double sided signals and their ROC

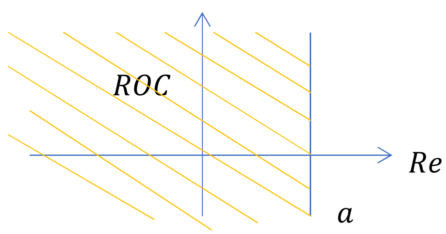


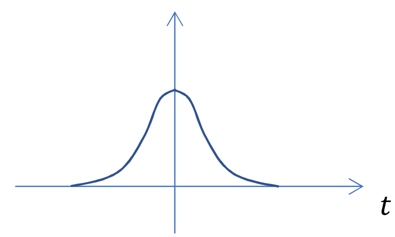




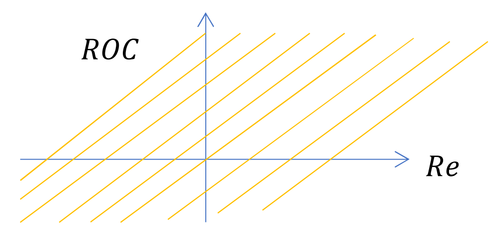








Bounded signal in time with finite values



Converge for all s

Laplace Transform & ROC

Example:

Find $x(t)$ for the following $X(s)$.

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3}, \quad ROC = \{s| \ Re(s) > -2\}$$

$$x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

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Laplace Transform & ROC

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Find $x(t)$ for the following $X(s)$.

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3}, \quad ROC = \{s| \ \ Re(s) > -2\}$$

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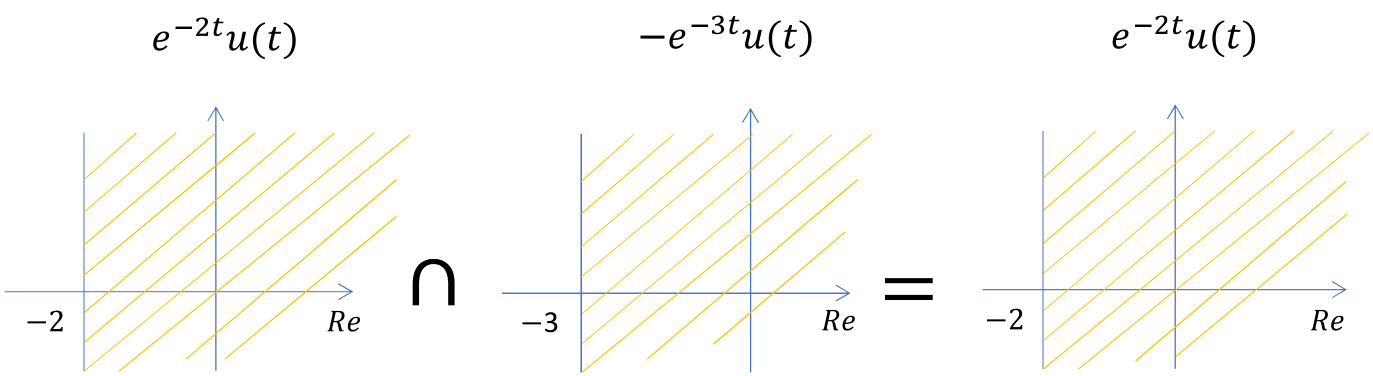

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Laplace Transform & ROC

Example:

Find $x(t)$ for the following $X(s)$.

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3}, \quad ROC = \{s| -3 < Re(s) < -2\}$$

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Laplace Transform & ROC

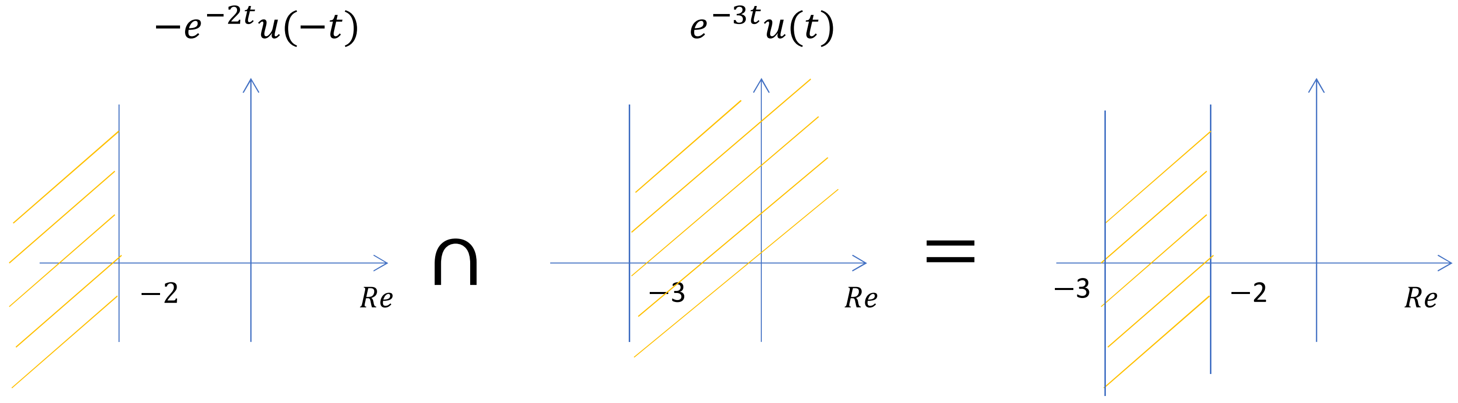
Example:

Find $x(t)$ for the following $X(s)$.

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3}, \quad ROC = \{s| -3 < Re(s) < -2\}$$

Signal has to be two-sided

$$x(t) = -e^{-2t}u(-t) + e^{-3t}u(t)$$



Without ROC there are more than one option for the inverse of the Laplace transform.

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Laplace Transform & ROC

Example:
Find possible $x(t)$s with the following Laplace transform:

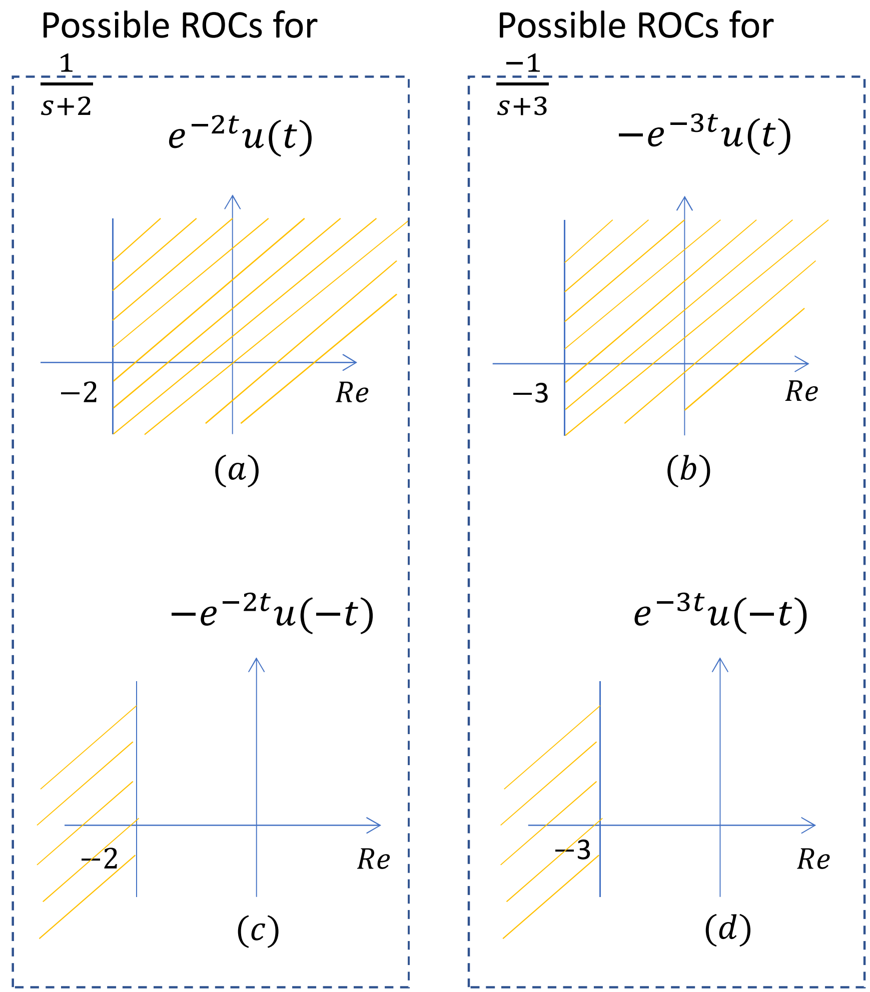
$$X(s) = \frac{1}{(s+2)(s+3)}$$

Use P.F.E

$$X(s) = \frac{a}{s+2} + \frac{b}{s+3} = \frac{1}{(s+2)(s+3)}$$

$$a(s+3) + b(s+2) = 1 \rightarrow \begin{cases} \text{set } s = -3 \Rightarrow b = -1 \\ \text{set } s = -2 \Rightarrow a = 1 \end{cases}$$

$$X(s) = \frac{1}{s+2} + \frac{-1}{s+3}$$



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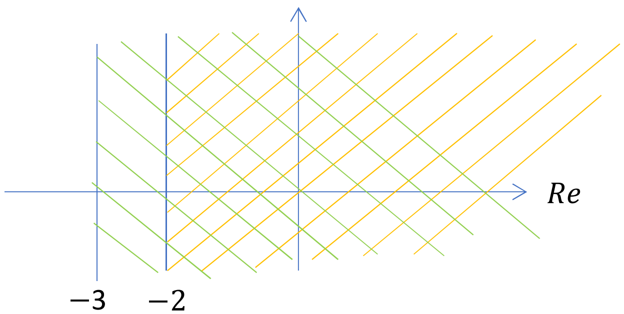
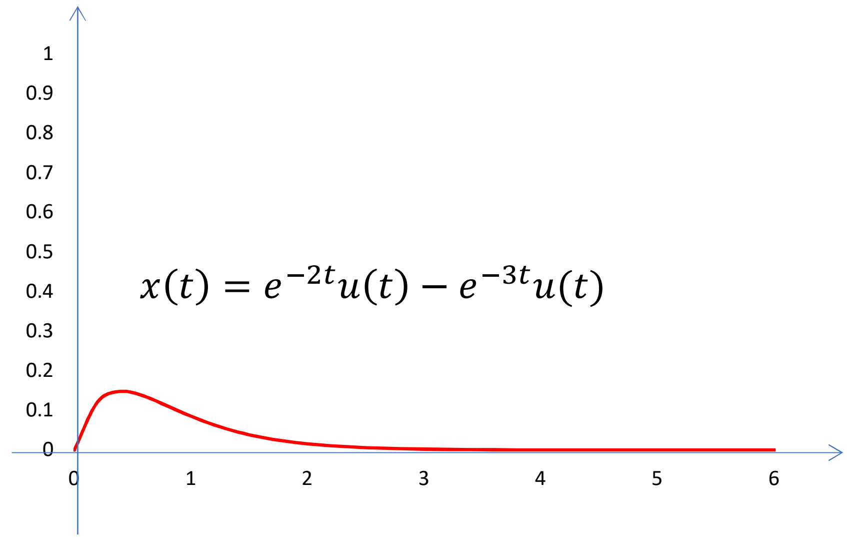
Laplace Transform & ROC

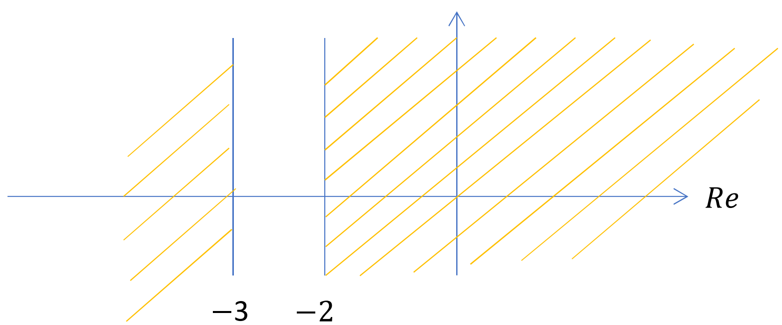
Case 1: (a) & (b):

Intersection: $Re\{s\} > -2$





Case 2: (a) & (d):



No intersection, so this is a signal that has no Laplace transform!

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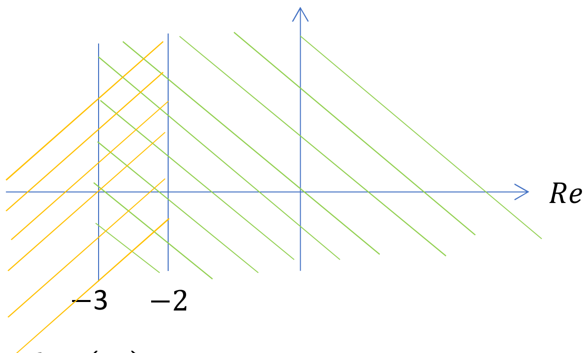
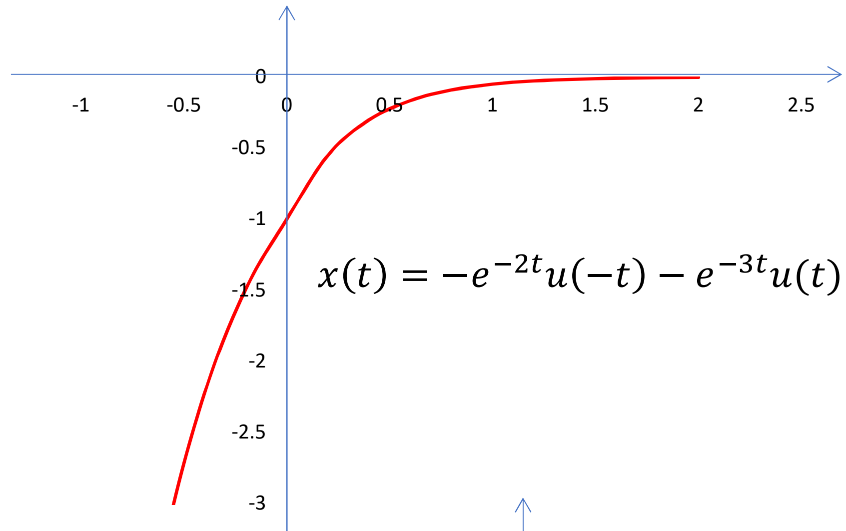
Laplace Transform & ROC

Case 3: (b) & (c):

Intersection: $-3 < Re\{s\} < -2$

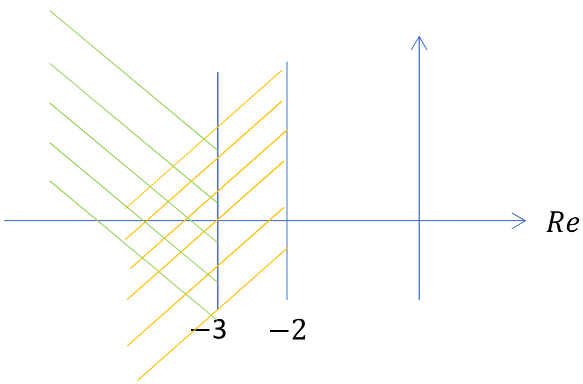


$$x(t) = -e^{-2t}u(-t) - e^{-3t}u(t)$$

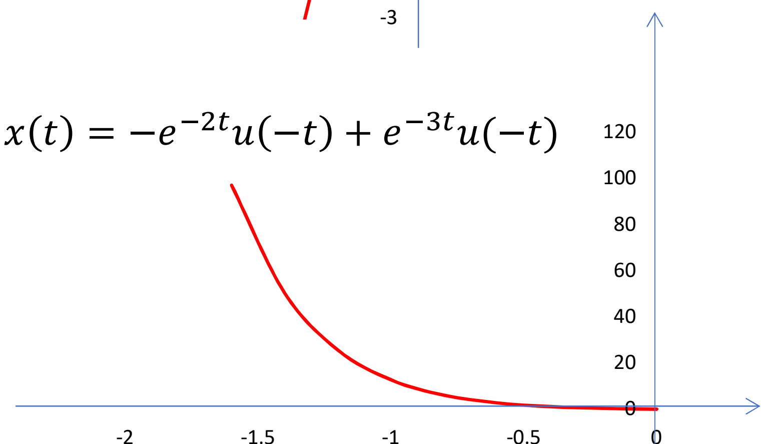


Case 4: (c) & (d):

Intersection: $Re\{s\} < -3$



$$x(t) = -e^{-2t}u(-t) + e^{-3t}u(-t)$$



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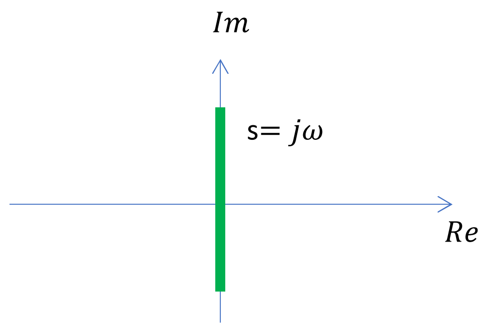
Laplace Transform & Fourier Transform

FT is a Laplace transform that is calculated for $s = j\omega$.



Therefore a signal has FT only if ROC includes the $j\omega$ axis! Therefore, **Not all signals have FT.**

Question: Go back to the previous example and indicate which signal has FT?

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Laplace Transform & ROC

Example:

Find $x(t)$ with the following $X(s)$:

$X(s) = \frac{2s^2+5}{s^2+3s+2}$ and $ROC: \quad Re\{s\} > -1$ (Right sided signal)

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Laplace Transform & ROC

Example:

Find $x(t)$ with the following $X(s)$:

$X(s) = \frac{2s^2+5}{s^2+3s+2}$ and $ROC: \quad Re\{s\} > -1$ (Right sided signal)

Solution:

$$X(s) = \frac{2(s^2+3s+2)+1-6s}{s^2+3s+2} = 2 + \frac{1-6s}{s^2+3s+2}$$

$$P.F.E\left(\frac{1-6s}{s^2+3s+2}\right) = \frac{a}{(s+1)} + \frac{b}{(s+2)} = \frac{1-6s}{(s+1)(s+2)}$$

$$a = \left.\frac{1-6s}{s+2}\right|_{s=-1} = 7$$

$$b = \left.\frac{1-6s}{s+1}\right|_{s=-2} = -13$$

$$x(t) = 2\delta(t) + 7e^{-t}u(t) - 13e^{-2t}u(t)$$

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Laplace Transform Properties

$$\text{Linearity} \quad ax_1(t) + bx_2(t) \rightarrow aX_1(s) + bX_2(s)$$

$$\text{Time Shift} \quad x(t - t_0) \rightarrow e^{-st_0}X(s)$$

$$\text{Frequency shift} \quad x(t)e^{s_0 t} \rightarrow X(s - s_0)$$

$$\text{Detivative} \quad \frac{dx(t)}{dt} \rightarrow sX(s)$$

$$\text{Higeher Order Derivative} \quad \frac{d^n x(t)}{dt^n} \rightarrow s^n X(s)$$

$$\int_{-\infty}^{t} x(t)dt \rightarrow \frac{1}{s}X(s)$$

$$\text{Scaling} \quad x(at) \rightarrow \frac{1}{a}X(\frac{s}{a})$$

$$x_1(t) * x_2(t) \rightarrow X_1(s)X_2(s)$$

$$x_1(t) \times x_2(t) \rightarrow \frac{1}{2\pi}X_1(s) * X_2(s)$$

Bilateral Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

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Laplace Transform Properties

Linearity $ax_1(t) + bx_2(t) \rightarrow aX_1(s) + bX_2(s)$

Time Shift $x(t - t_0) \rightarrow e^{-st_0}X(s)$

Frequency shift $x(t)e^{s_0 t} \rightarrow X(s - s_0)$

Detivative $\frac{dx(t)}{dt} \rightarrow sX(s) - x(0)$

Higeher Order Derivative $\frac{d^n x(t)}{dt^n} \rightarrow s^n X(s) - \sum_{k=1}^{n} s^{n-k} x^(k-1)(0)$

$$\int_{-\infty}^{t} x(t)dt \rightarrow \frac{1}{s}X(s)$$

Scaling $x(at) \rightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$

$$x_1(t) * x_2(t) \rightarrow X_1(s)X_2(s)$$

$$x_1(t) \times x_2(t) \rightarrow \frac{1}{2\pi}X_1(s) * X_2(s)$$

Initial value Theorem $x(0) = \lim_{s\rightarrow\infty} sX(s)$

Final value Theorem $x(\infty) = \lim_{s\rightarrow 0} sX(s)$

Bilateral Laplace Transform:

$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$

Unilateral Laplace Transform:

$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$

Unilateral Laplace is used for Causal signals and causal systems to deal with **initial conditions**.

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Laplace Transform Properties

Example:

$$h(t) = \delta(t-3)$$

$$H(s) = \int_{-\infty}^{\infty} \delta(t-3)e^{-ts}dt$$

$$= e^{-3s} \underbrace{\int_{-\infty}^{\infty} \delta(t-3)dt}_{1}$$

$$= e^{-3s}$$

Example:

$$X(s) = \frac{e^{-3s}}{s} = \frac{1}{s} \times e^{-3s}$$

$$x(t) = \underbrace{u(t)}_{\frac{1}{s}} * \underbrace{\delta(t-3)}_{e^{-3s}} = u(t-3)$$

Delay of $u(t)$ by 3 or
Integral of $\delta(t-3)$

Useful Laplace Transforms:

$$e^{-at}\cos(bt)u(t) \xrightarrow{\text{Laplace}} \frac{s+a}{(s+a)^2+b^2}$$

$$\delta(t) \xrightarrow{\text{Laplace}} 1$$

$$u(t) \xrightarrow{\text{Laplace}} \frac{1}{s}$$

$$tu(t) \xrightarrow{\text{Laplace}} \frac{1}{s^2}$$

Example:

$$X(s) = \frac{1}{s(s+5)} = \frac{a}{s} + \frac{b}{s+5}, \quad ROC = Re\{s\} > 0$$

$$a = \left.\frac{1}{s+5}\right|_{s=0} = \frac{1}{5}$$

$$b = \left.\frac{1}{s}\right|_{s=-5} = \frac{-1}{5}$$

$$x(t) = \frac{1}{5}u(t) - \frac{1}{5}e^{-5t}u(t)$$

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Laplace Transform Properties

Example:

$$x(t) = e^{-2t}\cos(\frac{\pi}{3}t)u(t)$$

$$= e^{-2t}\left(\frac{e^{j\frac{\pi}{3}t} + e^{-j\frac{\pi}{3}t}}{2}\right)u(t)$$

$$= \underbrace{\frac{e^{j\frac{\pi}{3}t}}{2}}_{\frac{1}{2}e^{s_0 t}}\underbrace{e^{-2t}u(t)}_{\frac{1}{s+2}} + \frac{e^{-j\frac{\pi}{3}t}}{2}e^{-2t}u(t)$$

$$= \frac{1}{2}\frac{1}{(s - \underbrace{j\frac{\pi}{3}}_{s_0}) + 2} + \frac{1}{2}\frac{1}{(s + j\frac{\pi}{3}) + 2}$$

$$= \frac{1}{2}\left[\frac{1}{(s - j\frac{\pi}{3}) + 2} + \frac{1}{(s + j\frac{\pi}{3}) + 2}\right]$$

$$= \frac{1}{2}\left[\frac{s + j\frac{\pi}{2} + 2 + s - j\frac{\pi}{2} + 2}{\left[(s - j\frac{\pi}{3}) + 2\right]\left[(s + j\frac{\pi}{3}) + 2\right]}\right]$$

$$= \frac{1}{2}\left[\frac{2s + 4}{\left[(s + 2) - j\frac{\pi}{3}\right]\left[(s + 2) + j\frac{\pi}{3}\right]}\right]$$

$$= \frac{s + 2}{(s + 2)^2 + (\frac{\pi}{9})^2} = \frac{s + 2}{s^2 + 4s + (4 + \frac{\pi^2}{9})}$$

Alternatively use:

$$e^{s_0 t}u(t),\ s_0 = \alpha_0 + j\omega_0 \rightarrow X_1(s) = \frac{1}{s - s_0}, ROC = Re\{s\} > \alpha_0$$

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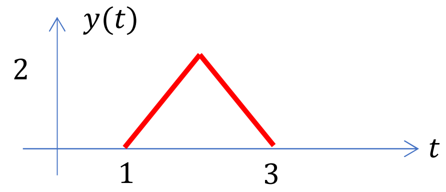
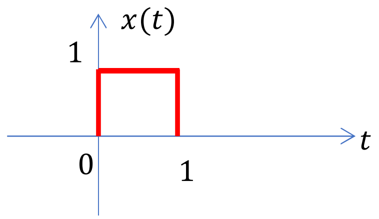
Laplace Transform Properties

Example:

Write $Y(s)$ as a function of $X(s)$.





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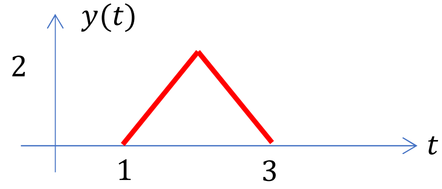
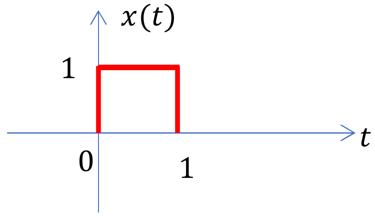
Laplace Transform Properties

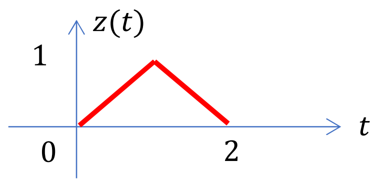
Example:

Write $Y(s)$ as a function of $X(s)$.





Solution:



$$x(t) * x(t) = z(t) \Rightarrow X(s) \times X(s) = Z(s)$$

$$2z(t-1) = y(t) \Rightarrow Y(z) = 2e^{-s}Z(s) = 2e^{-s}X^2(s)$$

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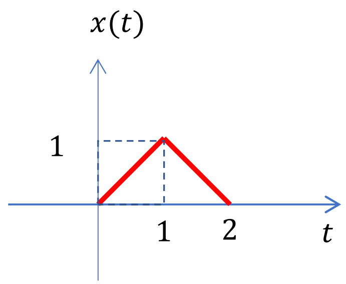
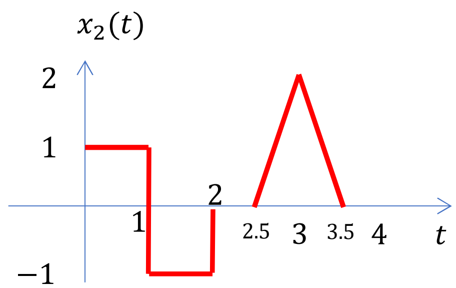
Laplace Transform Properties

Example: Write Laplace transform of $x_2(t)$ as a function of $X(s)$.





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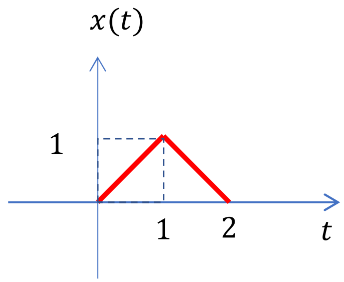
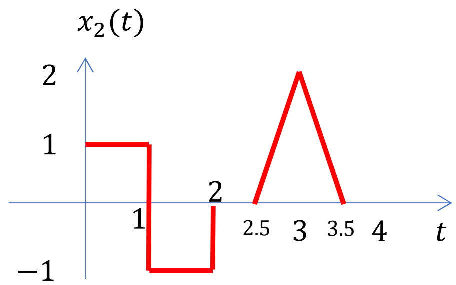
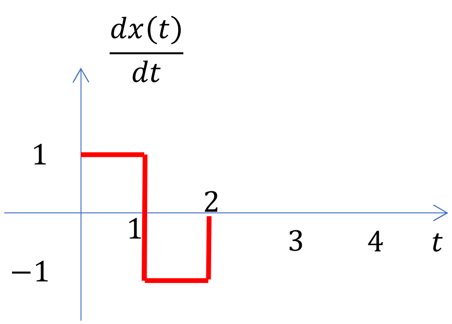
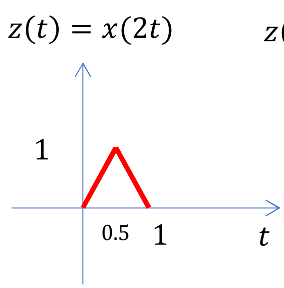
Laplace Transform Properties

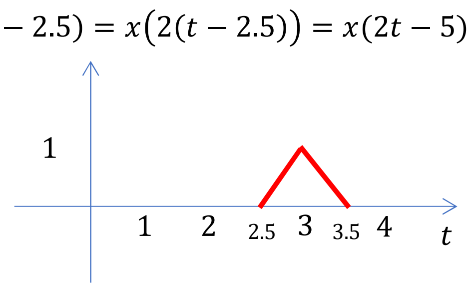
Example: Write Laplace transform of $x_2(t)$ as a function of $X(s)$.











$$
\begin{aligned}
x_2(t) &= \frac{dx(t)}{dt} + 2x(2t-5) \\
X_2(s) &= sX(s) + 2\mathscr{L}\left(x(2t-5)\right) \\
&= sX(s) + 2\mathscr{L}\left(z(t-2.5)\right) \\
&= sX(s) + 2e^{-2.5s}\mathscr{L}\left(z(t)\right) \\
&= sX(s) + 2e^{-2.5s}\mathscr{L}\left(x(2t)\right) \\
&= sX(s) + 2e^{-2.5s}\frac{1}{2}X(\frac{s}{2})
\end{aligned}
$$

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Laplace Transform

Example:

The input and output of a casual LTI system respectively are: $x(t) = e^{-2t}u(t)$ and $y(t) = te^{-t}u(t)$. Find $H(s)$, Laplace transform of the impulse response $h(t)$ and show its ROC.

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Laplace Transform

Example:
The input and output of a casual LTI system respectively are: $x(t) = e^{-2t}u(t)$ and $y(t) = te^{-t}u(t)$. Find $H(s)$, Laplace transform of the impulse response $h(t)$ and show its ROC.

Solution:

Laplace of impulse response of an LTI system is also known as **Transfer Function** of the system.

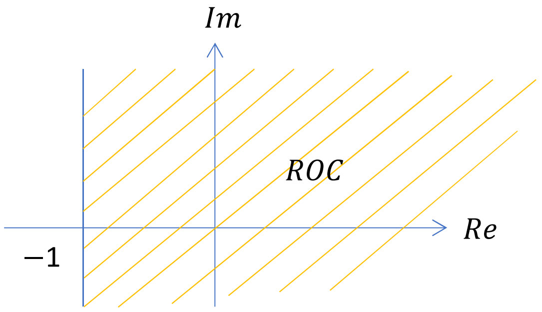
$$y(t) = x(t) * h(t) \rightarrow Y(s) = X(s)H(s) \rightarrow H(s) = \frac{Y(s)}{X(s)}$$

$$x(t) = e^{-2t}u(t) \rightarrow X(s) = \frac{1}{s+2}$$

$$y(t) = te^{-t}u(t) \rightarrow Y(s) = \frac{1}{(s+1)^2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{(s+1)^2}}{\frac{1}{(s+2)}} = \frac{s+2}{(s+1)^2}$$

Since the LTI system is casual, $h(t)$ is right hand signal, therefore ROC: $Re\{s\} > -1$



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Laplace Transform

Example:

LTI system has transform $H(s) = \frac{s-1}{s+1}$, what is $y(t)$ output of this system to input with Laplace transform $X(s) = \frac{s}{s+1}$.

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Laplace Transform

Example:
LTI system has transform $H(s) = \frac{s-1}{s+1}$, what is $y(t)$ output of this system to input with Laplace transform $X(s) = \frac{s}{s+1}$.

Solution:

$$Y(s) = H(s) \times X(s) = \frac{s-1}{s+1} \times \frac{s}{s+1} = \frac{s(s-1)}{(s+1)^2}$$

$$= \frac{s^2 - s}{s^2 + 2s + 1} = 1 + \frac{-3s-1}{s^2+2s+1} = 1 + \frac{a}{s+1} + \frac{b}{(s+1)^2}$$

$$b = (s+1)^2 Y(s)\Big|_{s=-1} = 2$$

$$a = \frac{d}{ds}\left((s+1)^2 Y(s)\right)\Big|_{s=-1} = (2s-1)\Big|_{s=-1} = -3$$

$$\frac{1}{(s+1)^2} \xrightarrow{IL} te^{-t}u(t)$$

$$y(t) = \delta(t) - 3e^{-t}u(t) + 2te^{-t}u(t)$$

Another method:

$$Y(s) = \frac{s^2-s}{(s+1)^2} = \underbrace{\frac{s^2}{(s+1)^2}}_{\left(\frac{d^2}{dt^2}(te^{-t}u(t))\right)} - \underbrace{\frac{s}{(s+1)^2}}_{\left(\frac{d}{dt}(te^{-t}u(t))\right)}$$

$$y(t) = \frac{d}{dt}[(e^{-t} - te^{-t})u(t)] - te^{-t}\delta(t) + (e^{-t} - te^{-t})u(t)$$
$$= \delta(t) + (2te^{-t} - 3e^{-t})u(t)$$

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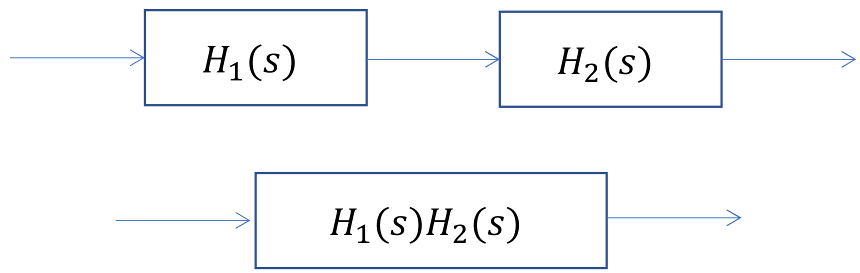
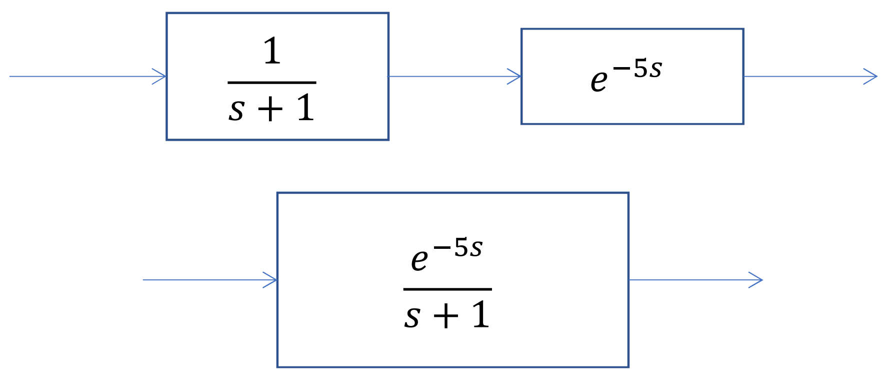
Laplace Transform

Cascade



Example:



$$H(s) = \frac{e^{-5s}}{s+1} \rightarrow h(t) = e^{-(t-5)}u(t-5)$$

$$Z(s) = \frac{1}{s+1} \rightarrow z(t) = e^{-t}u(t)$$

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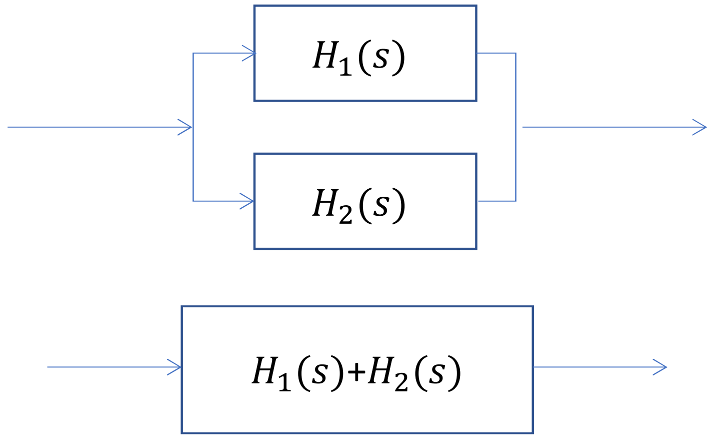
Laplace Transform

Parallel



Example:

$$H_1(s) = \frac{1}{s+2-4j}, \quad H_2(s) = \frac{1}{s+2+4j}$$

$$H(s) = \frac{1}{s+2-4j} + \frac{1}{s+2+4j} = \frac{s+2+4j+s+2-4j}{(s+2)^2-(4j)^2} = \frac{2s+4}{(s+2)^2+16}$$

$$e^{-at}\cos(bt)u(t) \xrightarrow{L} \tfrac{s+a}{(s+a)^2+b^2}$$

$$H(s) = \frac{2(s+2)}{(s+2)^2+4^2} \xrightarrow{\text{Inverse Laplace}} 2e^{-2t}\cos(4t)u(t)$$

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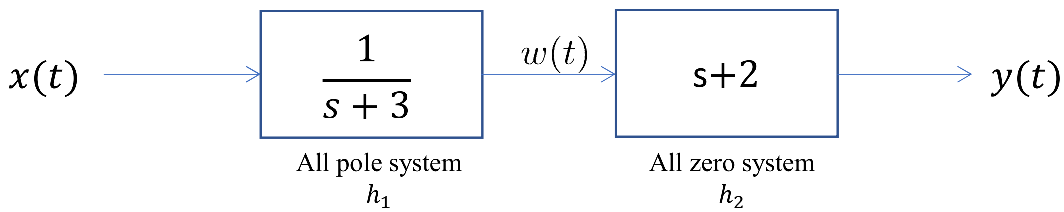
Laplace Transform

Example with zero & pole:

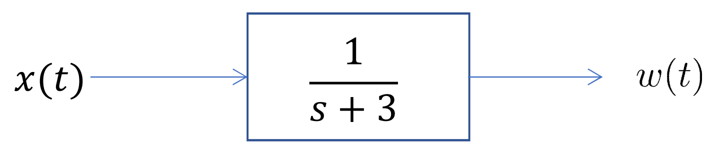
$$H(s) = \frac{s+2}{s+3}$$



$h_1(t) = e^{-t}u(t)$:

$$\frac{X(s)}{s+3} = W(s)$$

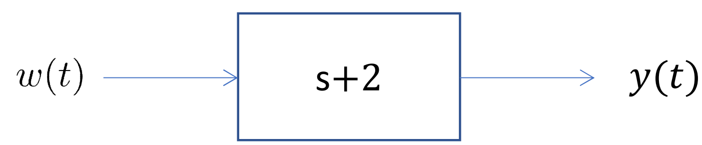
$$x(t) = \frac{d}{dt}w(t) + 3w(t)$$



$h_2(t) = \delta'(t) + 2\delta(t)$:

$$Y(s) = (s+2)W(s))$$

$$y(t) = \frac{d}{dt}w(t) + 2w(t)$$



$$h(t) = h_1(t) * h_2(t) = e^{-t}u(t) * (2\delta(t) + \delta'(t)) = 2e^{-t}u(t) + \frac{d}{dt}(e^{-t}u(t))$$

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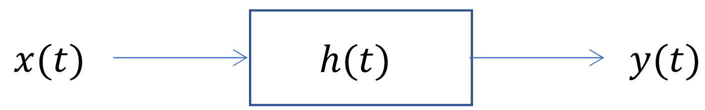
Laplace Transform



$$x(t) = e^{s_0 t} \qquad\qquad y(t) = H(s_0)e^{s_0 t}$$

Only for s_0s that are in ROC of H(s), otherwise the output is infinity!

Example:

$$x(t) = C \;\; \text{(Constant)} \Rightarrow y(t) = C \times H(j0)$$

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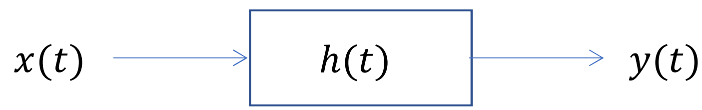
Laplace Transform



$x(t) = e^{s_0 t}$

$y(t) = H(s_0)e^{s_0 t}$

Only for s_0s that are in ROC of H(s), otherwise the output is infinity!

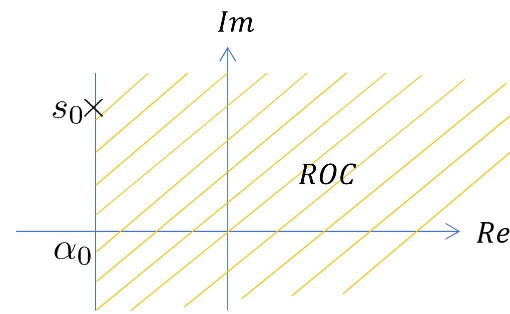
Note that $e^{s_0 t}$s are eigenfunctions of Laplace Transform!.

Difference between $e^{s_0 t}$ and its causal part $e^{s_0 t}u(t)$, $s_0 = \alpha_0 + j\omega_0$:

$$e^{s_0 t} \xrightarrow{\text{Laplace}} \delta(s - s_0)$$

$$ROC = \{s |\ Re\{s\} = Re\{s_0\} = \alpha_0\}$$

$$e^{s_0 t}u(t), \xrightarrow{\text{Laplace}} X_1(s) = \frac{1}{s - s_0}, ROC = \{s |\ Re\{s\} > \alpha_0\ \}$$



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