Signals and Systems I

Topic 5

Last Lecture

- System Classification

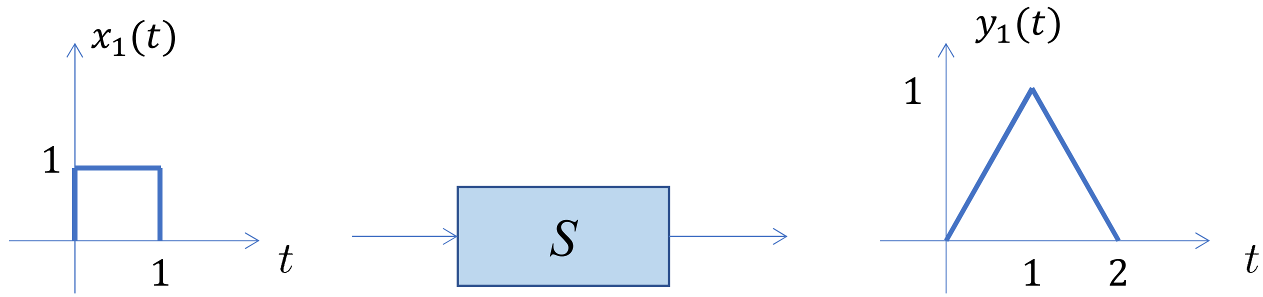
Today

- Linear Time-Invariant (LTI) Systems
- Impulse Response and its Importance for LTI Systems
- LTI Differential Equation (LTIDE) Systems
- Impulse Response of LTIDE Systems
- Convolution
- Initial Condition and LTI systems

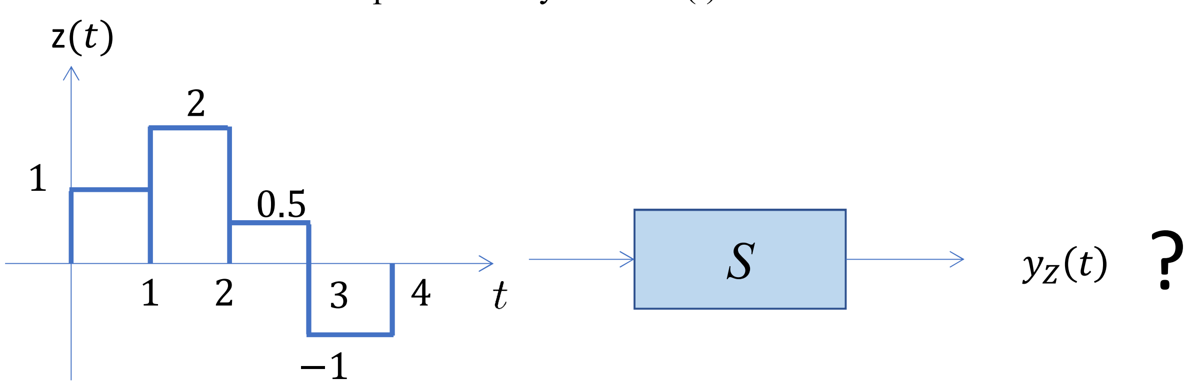
Why are Linear Time Invariant (LTI) Systems Important?

Linear Time Invariant (LTI) Systems are both Linear and Time Invariant (TI)!

Consider the following example:



What can be said about the output of this system to z(t).



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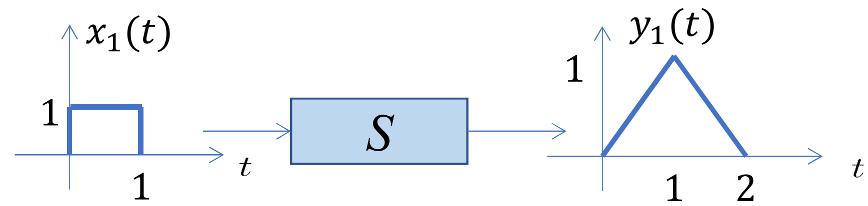
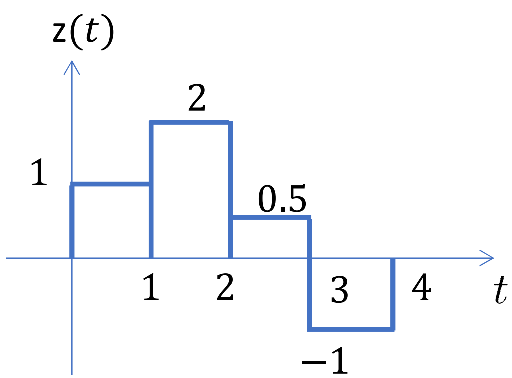
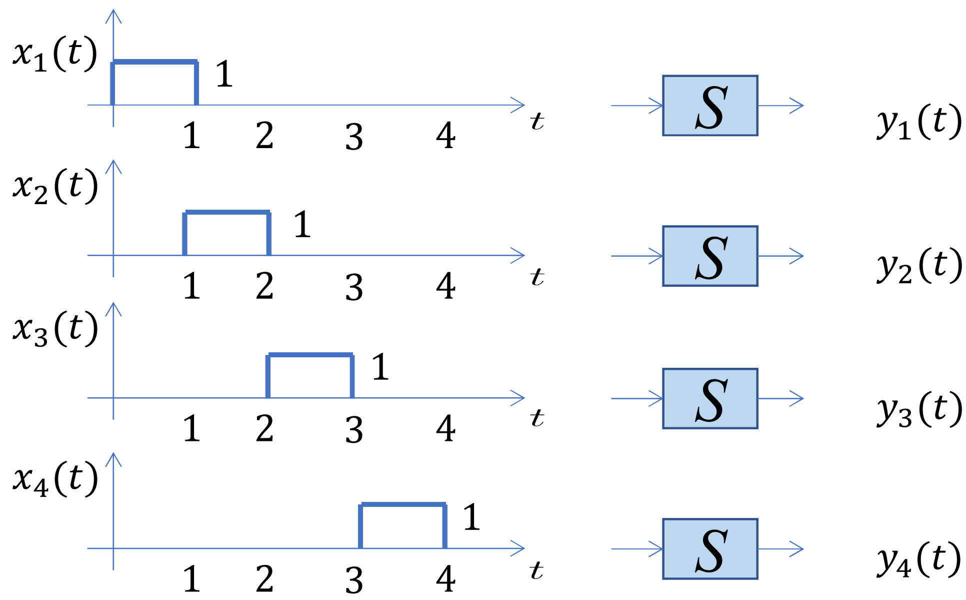
Why are Linear Time Invariant (LTI) Systems Important?



$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$





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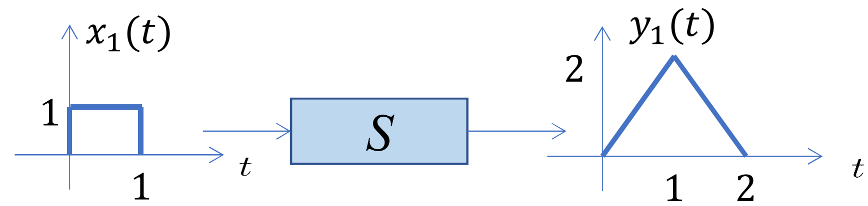
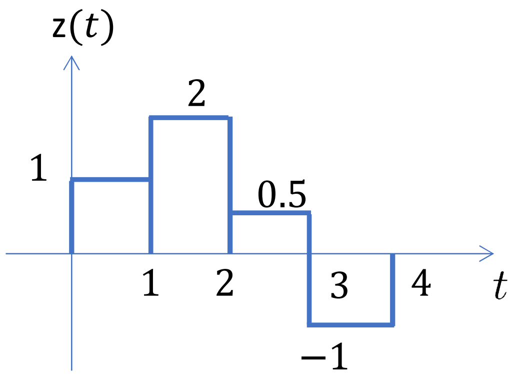
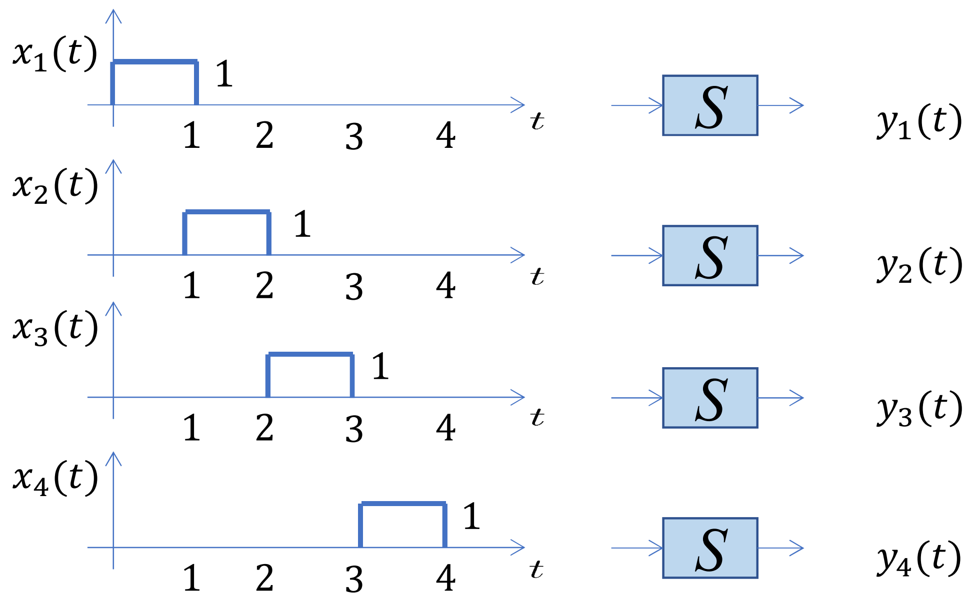
Why are Linear Time Invariant (LTI) Systems Important?



$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$





If the system is linear then:

$y_Z(t) = y_1(t) + 2y_2(t) + 0.5y_3(t) - y_4$

Can we find $y_2(t)$ and $y_3(t)$ and $y_4(t)$ only by using the available $y_1(t)$?

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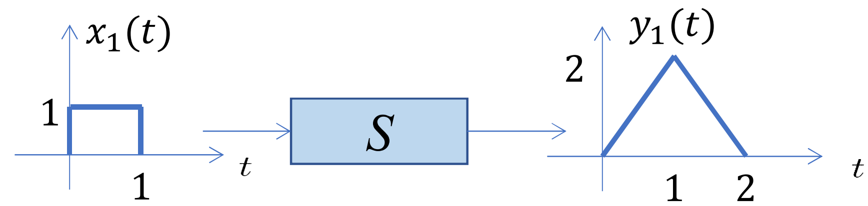
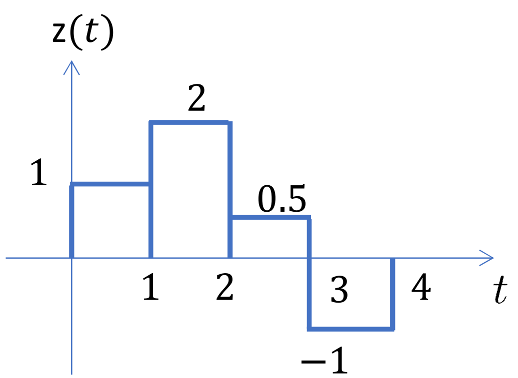
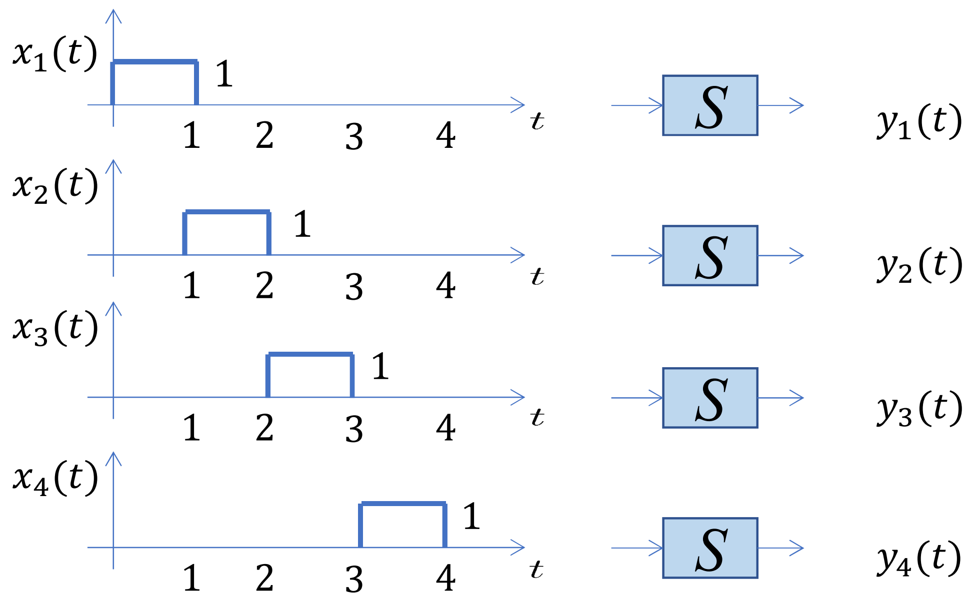
Why are Linear Time Invariant (LTI) Systems Important?



$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$





If the system is linear then:

$y_Z(t) = y_1(t) + 2y_2(t) + 0.5y_3(t) - y_4$

Can we find $y_2(t)$ and $y_3(t)$ and $y_4(t)$ only by using the available $y_1(t)$? Only if the system is also TI!

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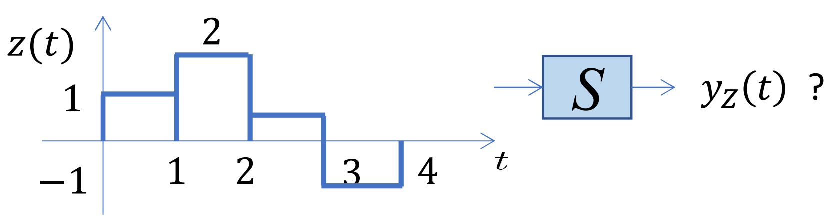
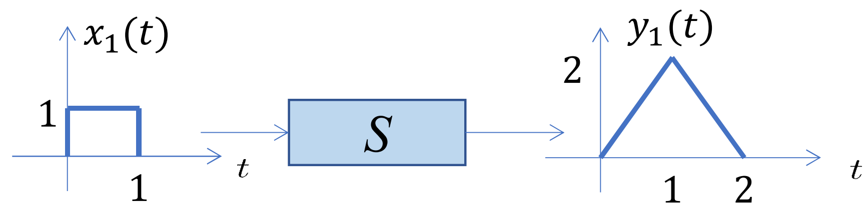
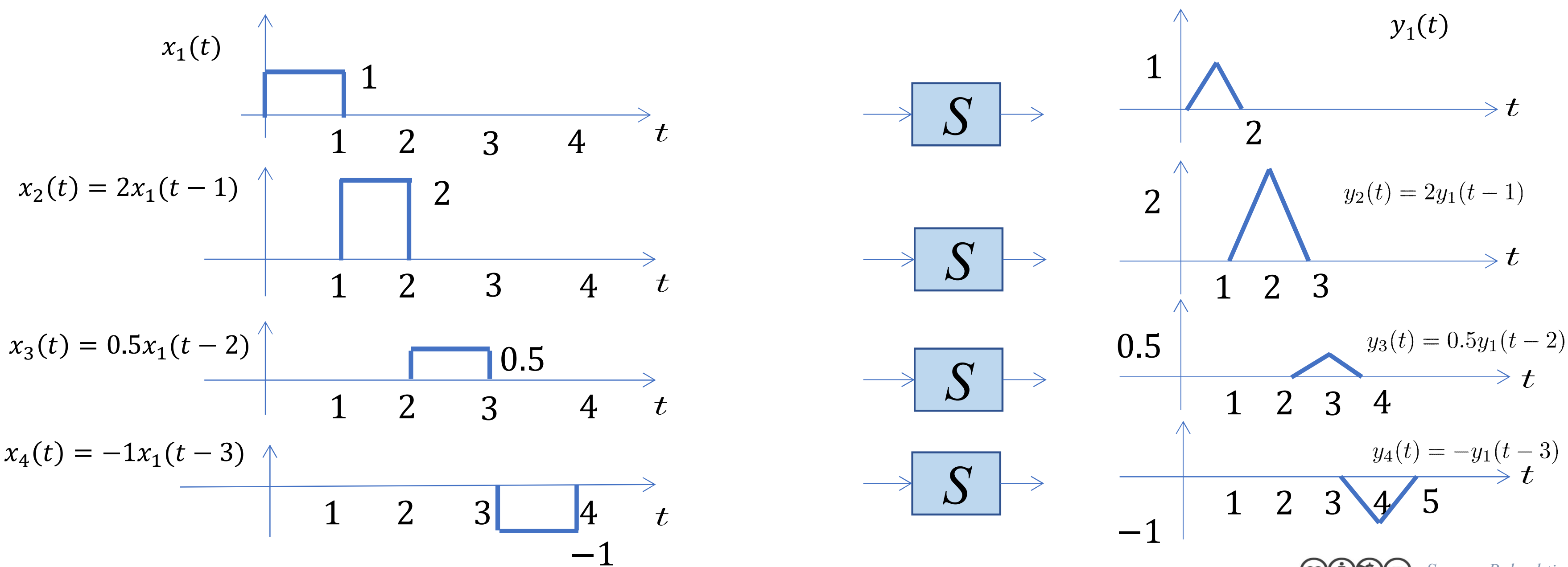
Why are Linear Time Invariant (LTI) Systems Important?





$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$

If the system is also TI, then we have:



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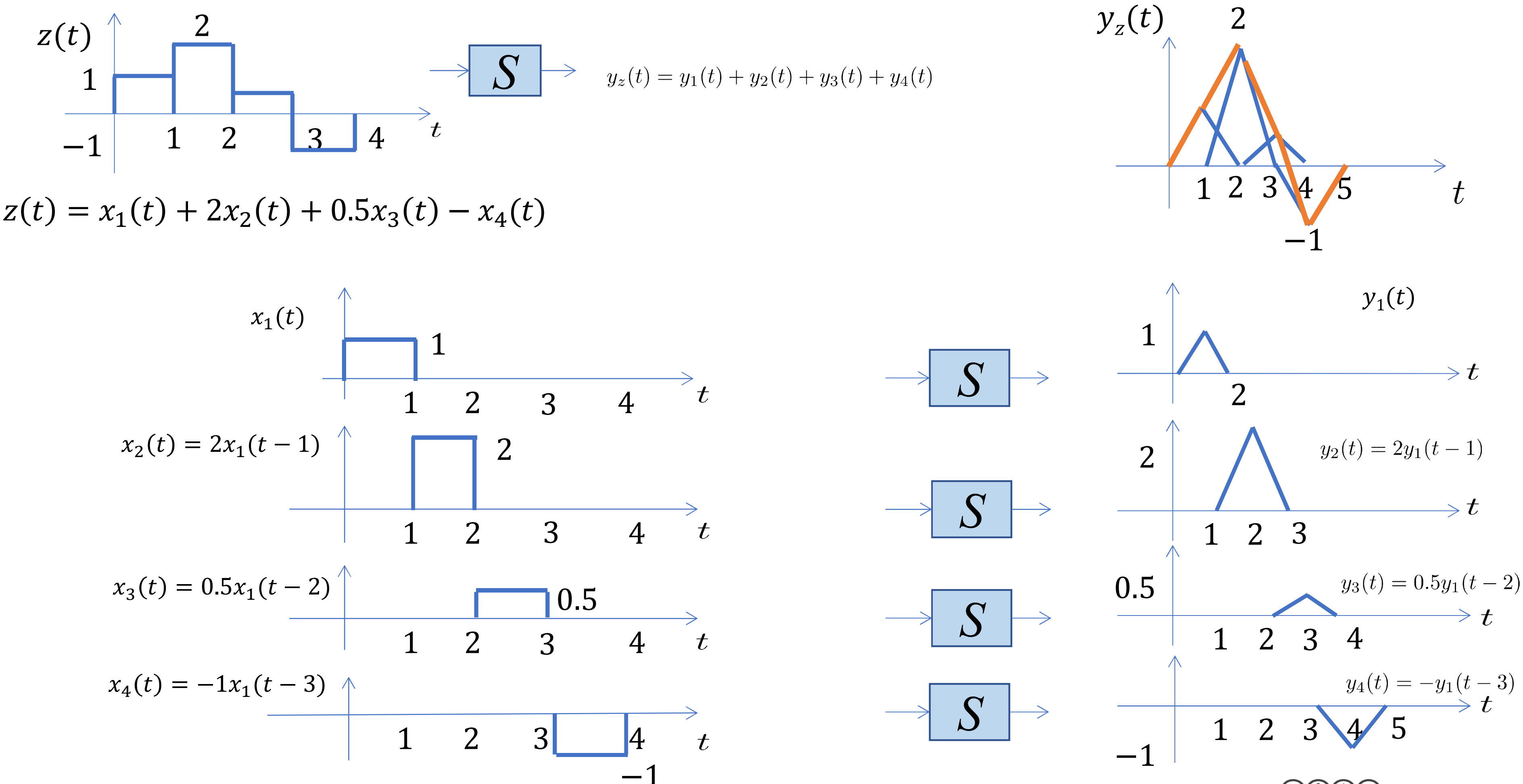
Why are Linear Time Invariant (LTI) Systems Important?



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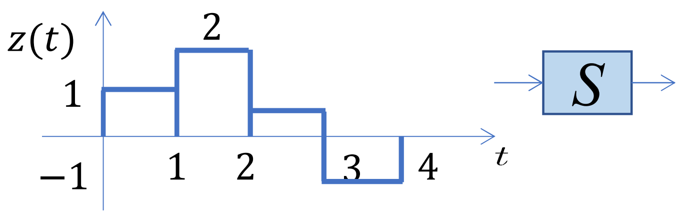
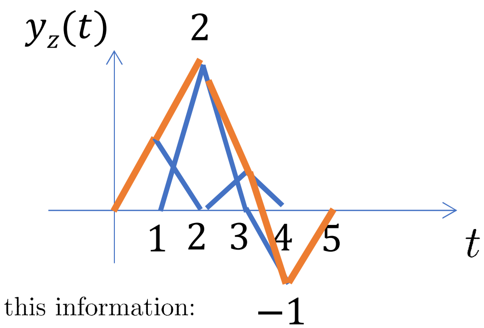
Why are Linear Time Invariant (LTI) Systems Important?

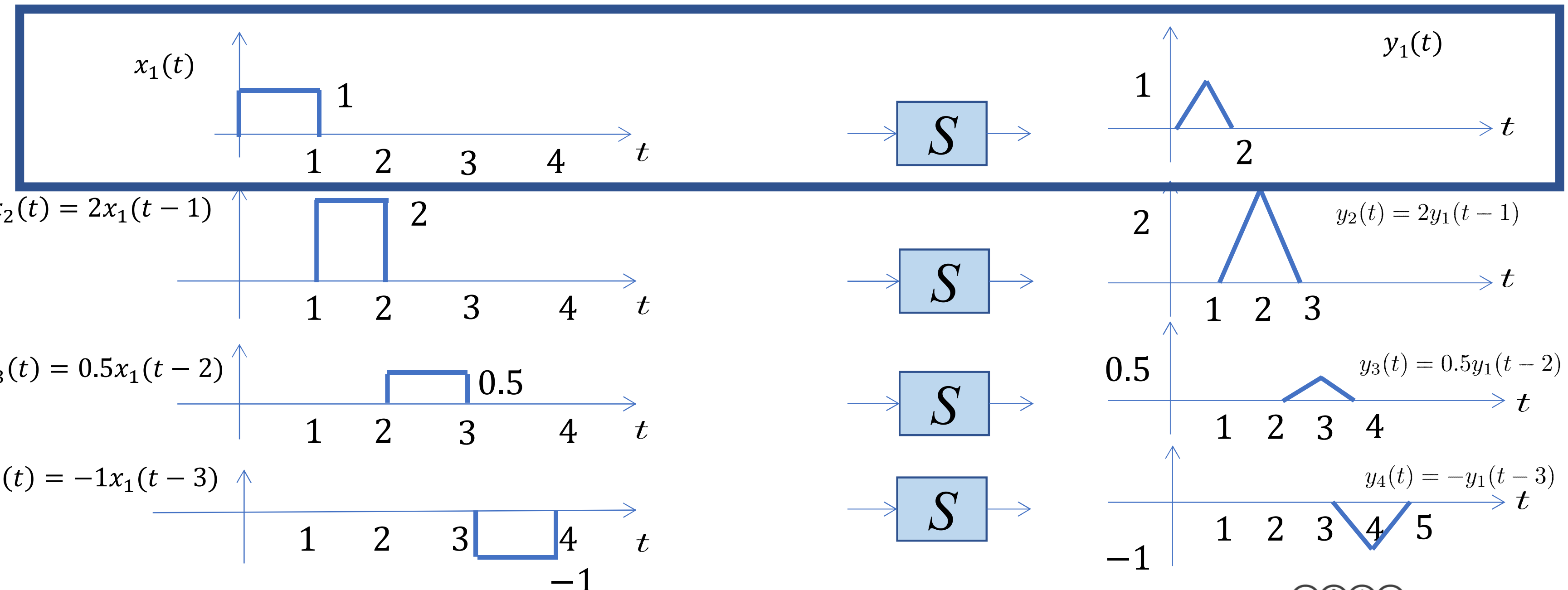


$y_z(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t)$



$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$

Since the system is LTI we found output $y_z(t)$ only with this information:



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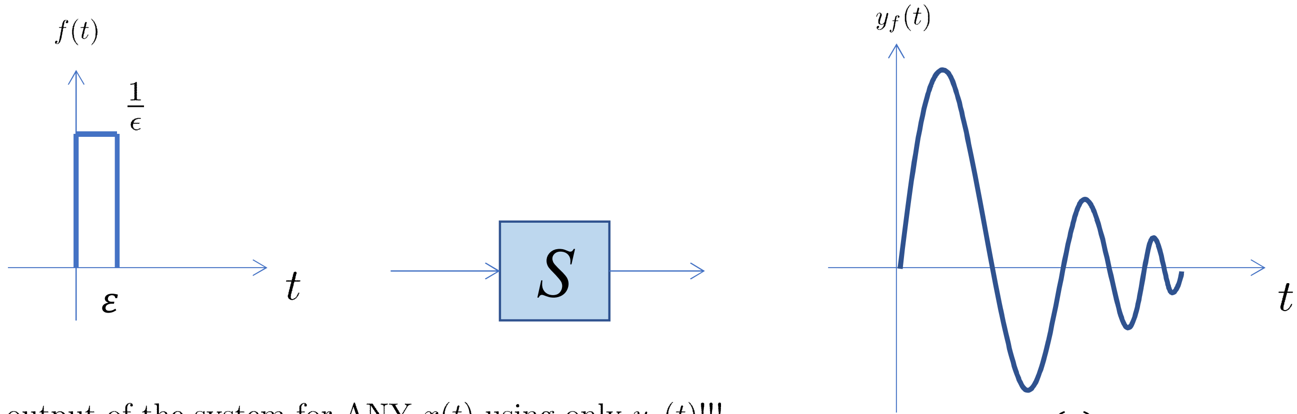
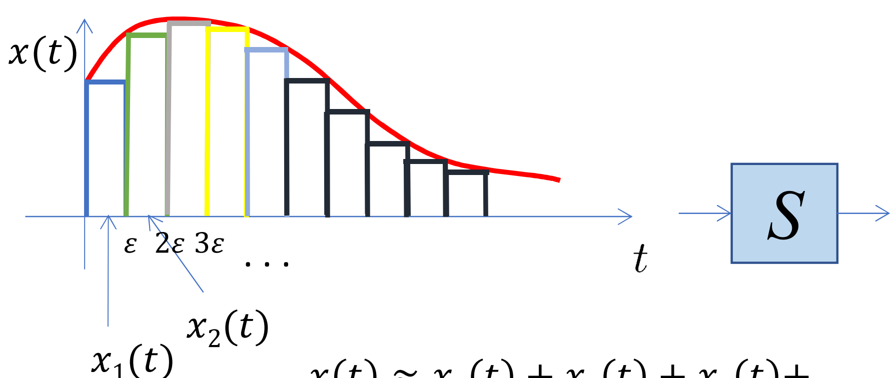
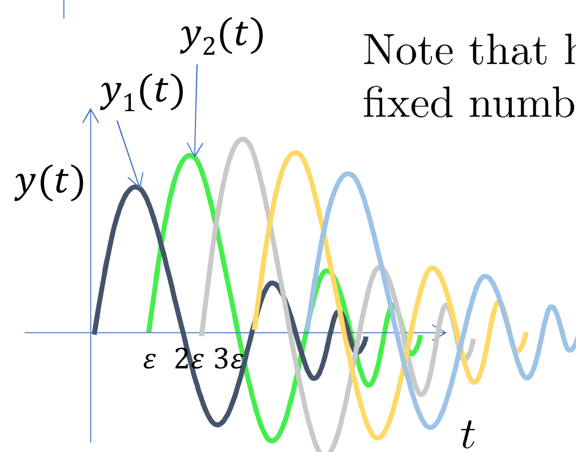
Why are Linear Time Invariant (LTI) Systems Important?

So for LTI system if we have the output of the system to $f(t)$



Then we can find the output of the system for ANY $x(t)$ using only $y_f(t)$!!!





Note that here $\epsilon x(n\epsilon)$ are fixed numbers (coefficients)

$$x(t) \approx x_1(t) + x_2(t) + x_3(t) + \dots$$
$$\approx \sum_n \varepsilon x(n\varepsilon) f(t - n\varepsilon)$$

$$y(t) \approx y_1(t) + y_2(t) + \cdots$$
$$= \sum_n \epsilon x(n\epsilon) y_f(t - n\epsilon)$$

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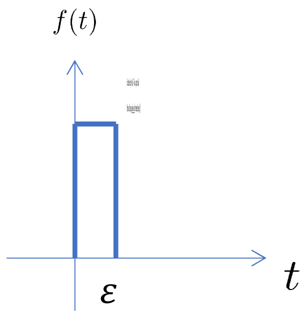
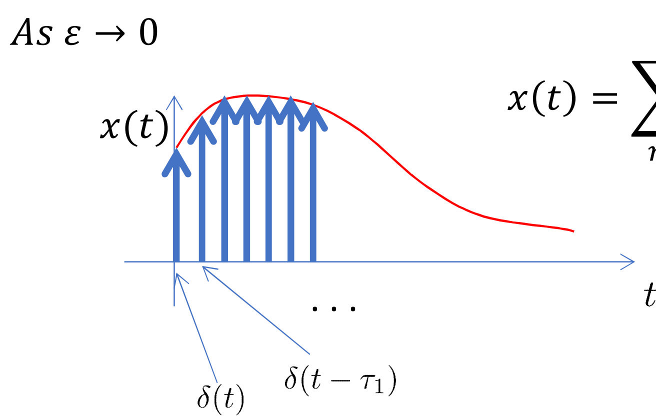
Linear Time Invariant (LTI) Systems and Impulse Response

As $\epsilon \to 0$, $f(t)$ becomes $\delta(t)$ and we have

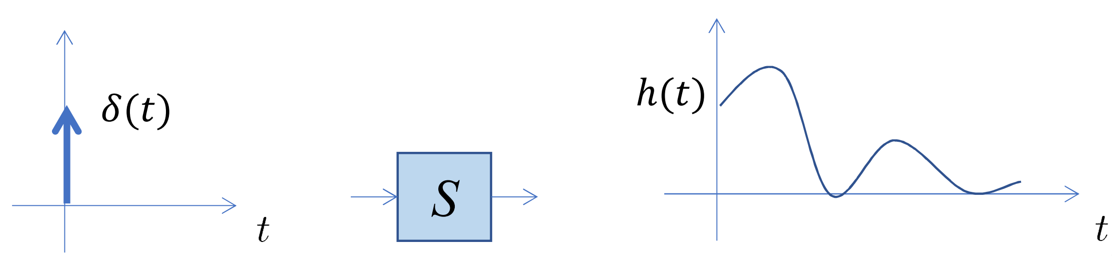




$$x(t) = \sum_n \varepsilon x(n\varepsilon) f(t - n\varepsilon) \to \quad x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

So if the output of linear system for all $\delta(t-T)$ is known , the output is available!

Definition: For **ALL** systems <u>impulse response, $h(t)$</u>, is the response of the system to <u>$\delta(t)$!</u>



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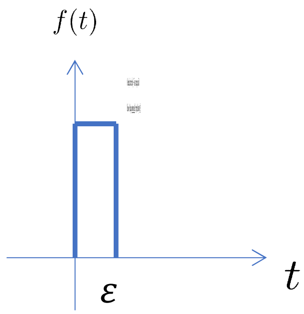
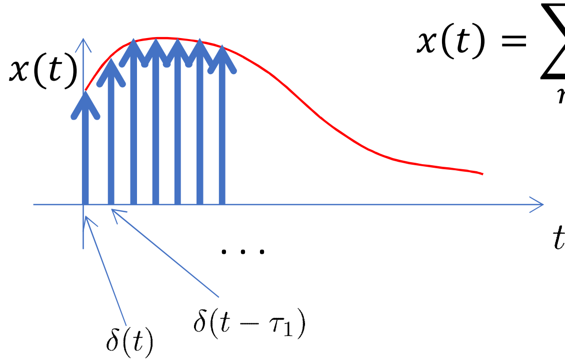
Linear Time Invariant (LTI) Systems and Impulse Response

As $\epsilon \to 0$, $f(t)$ becomes $\delta(t)$ and we have



As $\varepsilon \to 0$

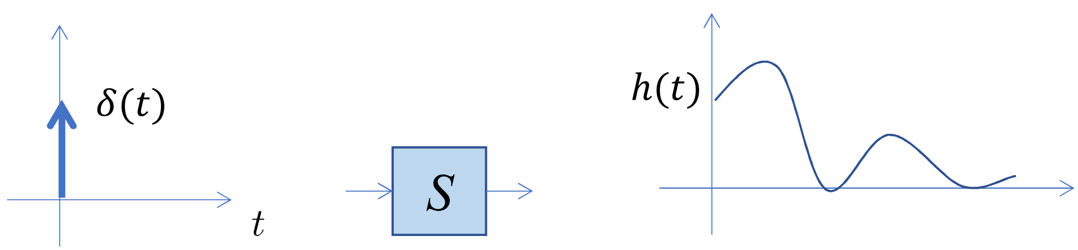


$$x(t) = \sum_{n} \varepsilon x(n\varepsilon) f(t - n\varepsilon) \rightarrow \quad x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Impulse response is very helpful and super important for LTI systems! Why?

So if the output of linear system for all $\delta(t-T)$ is known , the output is available!

Definition: For **ALL** systems <u>impulse response, $h(t)$</u>, is the response of the system to <u>$\delta(t)$!</u>



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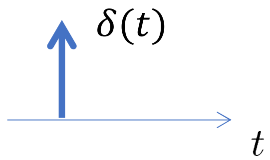
LTI Systems, Impulse Response and Convolution



$$\delta(t) \rightarrow \boxed{S} \rightarrow h(t)$$

Impulse response can be found for any system (even if it's not LTI)

However if the system is TI then:

$$\delta(t-\tau) \rightarrow \boxed{S} \rightarrow h(t-\tau)$$

And if the system is also Linear, then we have:

$$\sum x(\tau)\delta(t-\tau) \rightarrow \boxed{S} \rightarrow \sum x(\tau)h(t-\tau)$$

Here $x(\tau)$s are numbers like a and b in the linear combination of signals

And therefore

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \boxed{S} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

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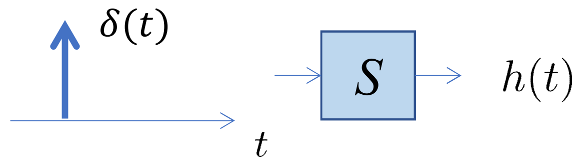
LTI Systems, Impulse Response and Convolution



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Here $x(\tau)$s are numbers like a and b in the linear combination of signals

And therefore

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \boxed{S} \rightarrow \boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

This operation is called Convolution

LTI systems are uniquely defined by their impulse response.

We can replace the LTI system with its impulse response that is a **signal**!

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Linear Time Invariant Differential Equation (LTIDE) systems

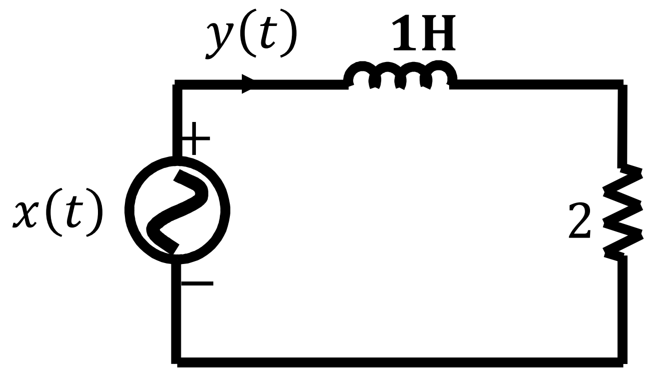
(important class of **LTI** systems)

Example:

$$x(t) = L\frac{dy}{dt} + Ry(t)$$

$$x(t) = \frac{dy}{dt} + 2y(t)$$

$$x(t) = (D+2)y(t)$$



Example:

$$x(t) - \frac{2d^2x(t)}{dt^2} + 3\frac{d^3x(t)}{dt^3} = y(t) + \frac{2dy}{dt}$$

$$x(t)(1 - 2D^2 + 3D^3) = (1+2D)y(t)$$

Linear combination of input and its higher order derivatives = Linear combination of output and it higher order derivatives

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General Form of Linear Time Invariant Differential Equation (LTIDE) systems

$$(D^N + a_1 D_{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_N) x(t)$$

N (highest derivative of output) is denoted as the ***order*** of the system.

For now we assume that $M \leq N$, (we discuss $M > N$ later).

Note that a_0 that is the coefficient associated with $D^N\, y(t)$ is ***one.*** If this is not the case first divide both sides so that a_0 is always one.

Example:

$$(D^2 + 5D + 6) y(t) = (3D^2 + D + 1) x(t)$$

$N = M = 2$ and $a_0 = 1, a_1 = 5, a_2 = 6, b_0 = 3, b_1 = 1, b_2 = 1$

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Impulse Response of LTIDE systems

Output of the causal system to imput $x(t) = \delta(t)$ is denoted by $h(t)$:

$$(D^N + a_1 D_{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_{N-M} D^M + b_{N-M+1}\, D^{M-1} + \cdots + b_N)\delta(t)$$

N is the order of the system (represent the number of poles of the system)

If $M = N$ then

$$h(t) = b_0 \delta(t) + (charactrsitic\ mode\ term\ for\ t > 0)$$

If $M < N$ then $b_0 = 0$,

$$h(t) = charactristic\ mode\ term\ for\ t > 0$$

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Impulse Response of LTIDE systems

Example :

$$(D^2 + 5D + 6)y(t) = (3D^2 + D + 1)x(t)$$

$M = N = 2, b_0 = 3$:

$$h(t) = 3\delta(t) + (char.mode\ term\ for\ t > 0\ )$$

Note: we are assuming that the system is also causal (will discuss this later)

Char. Equation (reminder): $\lambda^N + a_1\lambda^{N-1} + \cdots + a_n = 0$

Here: $\lambda^2 + 5\,\lambda + 6 = 0 \to \lambda_1 = -2\,, \lambda_2 = -3$

$$h(t) = 3\delta(t) + (c_1 e^{-2t} + c_2 e^{-3t})u(t)$$

The main challenge is now to find c_1 and c_2

Reminder: Char mode term (if there are no repeated roots for $t > 0$): $(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t})u(t)$

Char mode term (if there are no repeated roots) for $t > 0$: $\left(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right)u(t)$

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Impulse Response of LTIDE systems

Example 1: Find the impulse response to the following LTIDE:

Input $x(t)$ and output $y(t)$

$$\left(D^2+5D+6\right)y(t)=(D+1)\,x(t)$$

We replace $x(t)=\delta(t)$ so $y(t)=h(t)$ (A kick to the system!)

$$\left(D^2+5D+6\right)h(t)=(D+1)\,\delta(t)$$

$$h''+5h'+6h=\delta'+\delta$$

$N=2, M=1, b_0=0$

$\lambda^2+5\lambda+6=0\rightarrow\lambda_1=-2,\lambda_2=-3\rightarrow h(t)=\left(c_1e^{-2t}+c_2e^{-3t}\right)u(t)$

$h'(t)=(-2c_1e^{-2t}-3c_2e^{-3t})u(t)+(c_1e^{-2t}+c_2e^{-3t})\delta(t)$
$h'(t)=(-2c_1e^{-2t}-3c_2e^{-3t})u(t)+(c_1+c_2)\delta(t)$

$h''(t)=(4c_1e^{-2t}+9c_2e^{-3t})u(t)+(-2c_1e^{-2t}-3c_2e^{-3t})\delta(t)+(c_1+c_2)\delta'(t)$

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Impulse Response of LTIDE systems

$$\left(D^2 + 5D + 6\right) h(t) = (D+1)\,\delta(t)$$

So we build $(D^2 + 5D + 6)h(t)$ and set it equal to $(D+1)\delta(t) = \delta'(t) + \delta(t)$.

$$6 \times h(t) = 6 \times \left[(c_1 e^{-2t} + c_2 e^{-3t})u(t)\right]$$
$$5 \times h'(t) = 5 \times \left[(-2c_1 e^{-2t} - 3c_2 e^{-3t})u(t) + (c_1 + c_2)\delta(t)\right]$$
$$1 \times h''(t) = 1 \times \left[(4c_1 e^{-2t} + 9c_2 e^{-3t})u(t) + (-2c_1 - 3c_2)\delta(t) + (c_1 + c_2)\delta'(t)\right]$$

$$6h(t) + 5h'(t) + h''(t) = 0 \times u(t) + (3c_1 + 2c_2)\delta(t) + (c_1 + c_2)\delta'(t)$$
$$\text{has to be the same as} = \delta(t) + \delta'(t)$$

$$\begin{cases} 3c_1 + 2c_2 & = 1 \\ c_1 + c_2 & = 1 \end{cases} \rightarrow \begin{cases} c_1 = -1 \\ c_2 = 2 \end{cases} \quad \rightarrow h(t) = (-e^{-2t} + 2e^{-3t})u(t)$$

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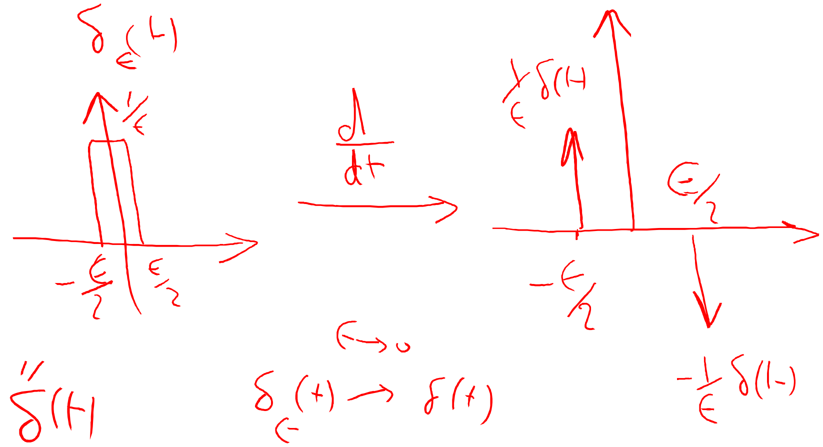
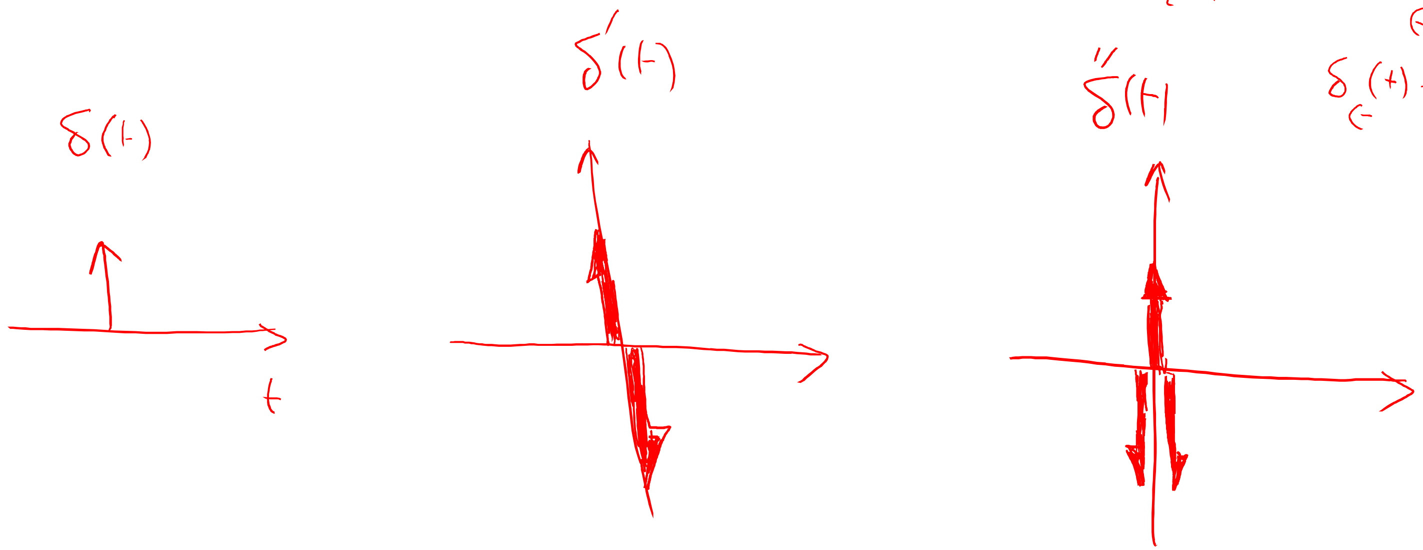
Impulse Response of LTIDE systems

$\delta(t)$ and its derivatives $(\delta'(t), \cdots)$





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Impulse Response of LTIDE systems

Example 2: Find the impulse response to the following system:

$$(D^2 + 3D + 2)y(t) = Dx(t) \qquad N = 2, M = 1, b_0 = 0$$

$$\lambda^2+3\lambda+2 = 0 \rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases} \rightarrow h(t) = \left(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right) u(t) = \left(c_1 e^{-t} + c_2 e^{-2t}\right) u(t)$$

So we have to have:

$$h''(t) + 3h'(t) + 2h(t) = \delta'(t)$$

$$\begin{aligned} h(t) &= \left(c_1 e^{-t} + c_2 e^{-2t}\right) u(t) \\ h'(t) &= \left(-c_1 e^{-t} - 2c_2 e^{-2t}\right) u(t) + (c_1 + c_2)\delta(t) \\ h''(t) &= \left(c_1 e^{-t} + 4c_2 e^{-2t}\right) u(t) + (-c_1 - 2c_2)\delta(t) + (c_1 + c_2)\delta'(t) \end{aligned}$$

We can write:

$$\begin{aligned} h''(t) + 3h'(t) + 2h(t) &= \delta'(t) \\ 0 \times u(t) + (2c_1 + c_2)\delta(t) + (c_1 + c_2)\delta'(t) &= \delta'(t) \end{aligned}$$

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Impulse Response of LTIDE systems

Example 2: Find the impulse response to the following system:

$$(D^2 + 3D + 2)y(t) = Dx(t) \qquad N = 2, M = 1, b_0 = 0$$

$$h''(t) + 3h'(t) + 2h(t) = \delta'(t)$$

$$h(t) = \left(c_1 e^{-t} + c_2 e^{-2t}\right) u(t)$$
$$h'(t) = \left(-c_1 e^{-t} - 2c_2 e^{-2t}\right) u(t) + (c_1 + c_2)\delta(t)$$
$$h''(t) = \left(c_1 e^{-t} + 4c_2 e^{-2t}\right) u(t) + (-c_1 - 2c_2)\delta(t) + (c_1 + c_2)\delta'(t)$$

We can write:

$$h''(t) + 3h'(t) + 2h(t) = \delta'(t)$$
$$0 \times u(t) + (2c_1 + c_2)\delta(t) + (c_1 + c_2)\delta'(t) = \delta'(t)$$

$$\begin{cases} 2c_1 + c_2 = 0 \\ c_1 + c_2 = 1 \end{cases} \rightarrow c_1 = -1, c_2 = 2 \rightarrow \boxed{h(t) = \left(-e^{-t} + 2e^{-2t}\right) u(t)}$$

Validate your answer

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Impulse Response of LTIDE systems

$$(D^2 + 3D + 2)y(t) = Dx(t) \qquad N = 2, M = 1, b_0 = 0$$

Here is how to validate your answer:

$$h(t) = \left(e^{-t} + 2e^{-2t}\right)u(t)$$
$$h'(t) = \left(e^{-t} - 4e^{-2t}\right)u(t) + \delta(t)$$
$$h''(t) = \left(-e^{-t} + 8e^{-2t}\right)u(t) + (-3)\delta(t) + \delta'(t)$$

$$2h(t) + 3h'(t) + h''(t) = 0 \times u(t) + (3-3)\delta(t) + \delta'(t)$$

Confirmed!

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Impulse Response of LTIDE systems

Example 3: Find the impulse response to the following system:

$$(D+2)y(t) = (3D+5)x(t) \qquad N = 1, M = 1, b_0 = 3$$

Solution:

$$x(t) = \delta(t) \to y(t) = h(t)$$

$$h(t) = b_0\delta(t) + (\text{Char. mode term for } t > 0)$$

$$\lambda + 2 = 0 \to \lambda = -2 \to h(t) = 3\delta(t) + ce^{-2t}u(t)$$

$$h'(t) = 3\delta'(t) - 2ce^{-2t}u(t) + c\delta(t)$$

$$h'(t) + 2h(t) = 3\delta'(t) + 5\delta(t)$$

$$3\delta'(t) + 0 \times u(t) + (c+6)\delta = 3\delta'(t) + 5\delta(t)$$

$$c + 6 = 5 \to c = -1$$

$$\boxed{h(t) = 3\delta(t) - e^{-2t}u(t)}$$

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Impulse Response of LTIDE systems

Example 4: Find the impulse response to the following system:

$$(D^2 + 2D + 1)y(t) = Dx(t) \qquad N = 2, M = 1, b_0 = 0$$

Solution:

$$h'' + 2h' + h = \delta'(t)$$

$$\text{Char. roots: } \lambda^2 + 2\lambda + 1 = 0 \rightarrow \lambda = -1, -1 \quad \text{(repeated roots)}$$

$$\begin{aligned} h(t) &= \left(c_1 e^{-t} + t c_2 e^{-t}\right) u(t) \\ h'(t) &= \left(-c_1 e^{-t} + (c_2 e^{-t} - t c_2 e^{-t})\right) u(t) + c_1 \delta(t) \\ h''(t) &= \left(c_1 e^{-t} - c_2 e^{-t} - c_2 e^{-t} + t c_2 e^{-t}\right) u(t) + \left(-c_1 + c_2\right) \delta(t) + c_1 \delta'(t) \end{aligned}$$

$$\begin{aligned} h''(t) + 2h'(t) + h(t) &= \delta'(t) \\ 0 \times u(t) + (c_1 + c_2)\delta(t) + c_1 \delta'(t) &= \delta'(t) \end{aligned}$$

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 = 1 \end{cases} \rightarrow \begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases} \rightarrow \boxed{h(t) = (1 - t)e^{-t} u(t)}$$

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Impulse Response of LTIDE systems

Example 5: Find the impulse response to the following system:

$$(D^2 + 1)y(t) = 2x(t) \qquad N = 2, M = 0, b_0 = 0$$

Solution:

Char. roots: $\lambda^2 + 1 = 0 \rightarrow \lambda = +j, -j$

$h(t) = (c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t})u(t)$

$$h(t) = (c_1 e^{jt} + c_2 e^{-jt})u(t)$$

$$h'(t) = (jc_1 e^{jt} - jc_2 e^{-jt})u(t) + (c_1 + c_2)\delta(t)$$

$$h''(t) = (-c_1 e^{jt} - c_2 e^{-jt})u(t) + (jc_1 - jc_2)\delta(t) + (c_1 + c_2)\delta'(t)$$

$$h''(t) + h(t) = 2\delta(t)$$

$$h''(t) + h(t) = 0 \times u(t) + (jc_1 - jc_2)\delta(t) + (c_1 + c_2)\delta'(t) = 2\delta(t)$$

$$\begin{cases} jc_1 - jc_2 = 2 \\ c_1 + c_2 = 0 \end{cases} \rightarrow \begin{cases} c_1 = -j \\ c_2 = j \end{cases} \rightarrow \boxed{h(t) = (-je^{jt} + je^{-jt})u(t) = 2sin(t)u(t)}$$

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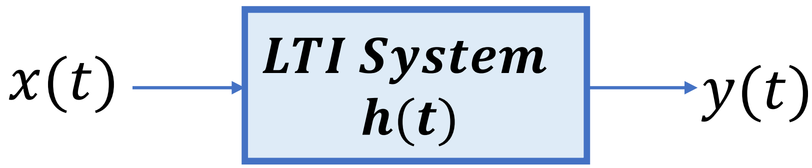
Convolution



The output, $y(t)$, of a LTI system with impulse response $h(t)$ and input $x(t)$ can be calculated using convolution as follows:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Note that in this equation for each value of t, variable τ is the independent variable inside the equation and t is the associated **delay**.

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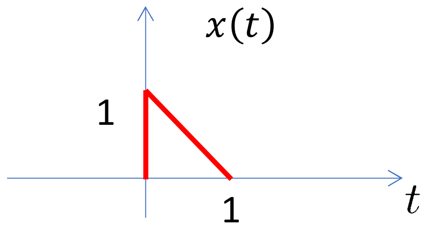
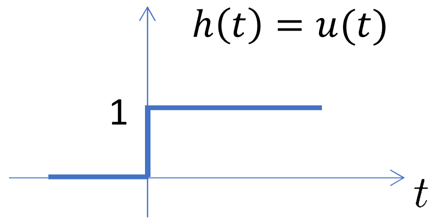
Convolution

Example: Find $y(t)$ output of the following system to $x(t)$





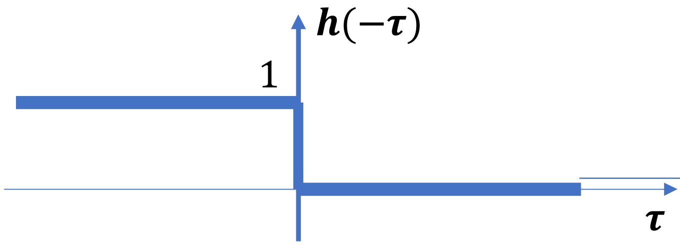
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

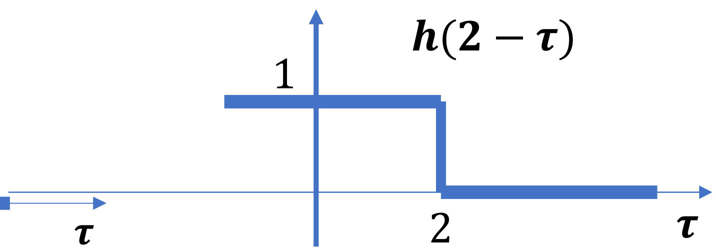
Solution:

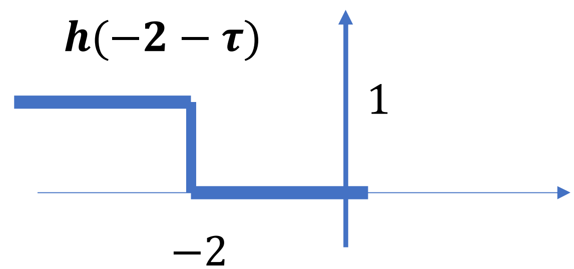
First we need to build $h(t-\tau)$ (here t is the delay and τ is the IV)

1- Flip $h(\tau)$ horizontally to build $h(-\tau)$.

2- two examples of $h(t-\tau)$ for $t = 2$ and $t = -2$.







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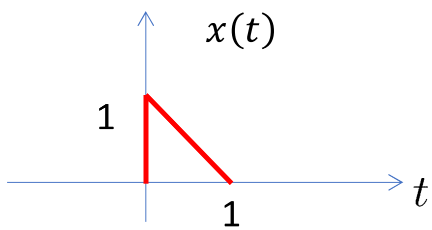
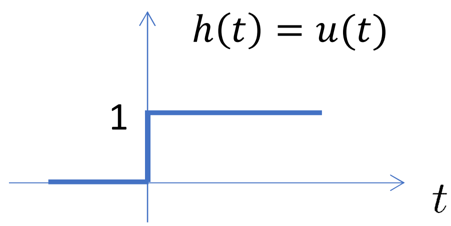
Convolution

Example: Find $y(t)$ output of the following system to $x(t)$





$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

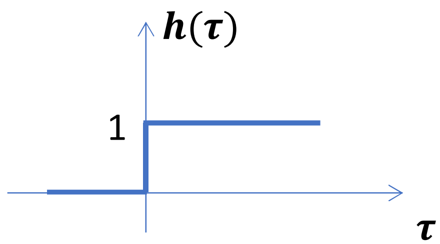
Solution:

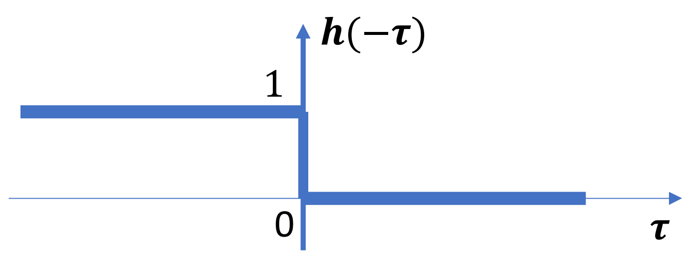
First we need to build $h(t-\tau)$ (here t is the delay and τ is the independent variable (IV))

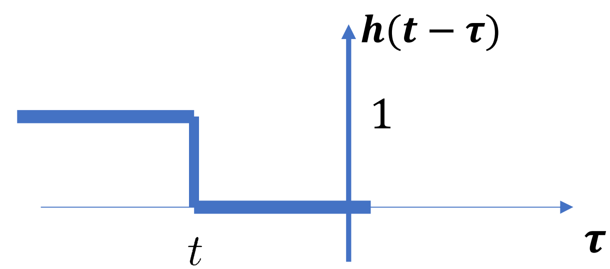
Step 1- Change the IV from t to τ.

Step 2- Flip $h(\tau)$ to build $h(-\tau)$.

3- Move the value $h(-\tau)$ at zero to t to built $h(t-\tau)$







next step is to find $x(\tau)h(t-\tau)$ for different values of t and its integral:

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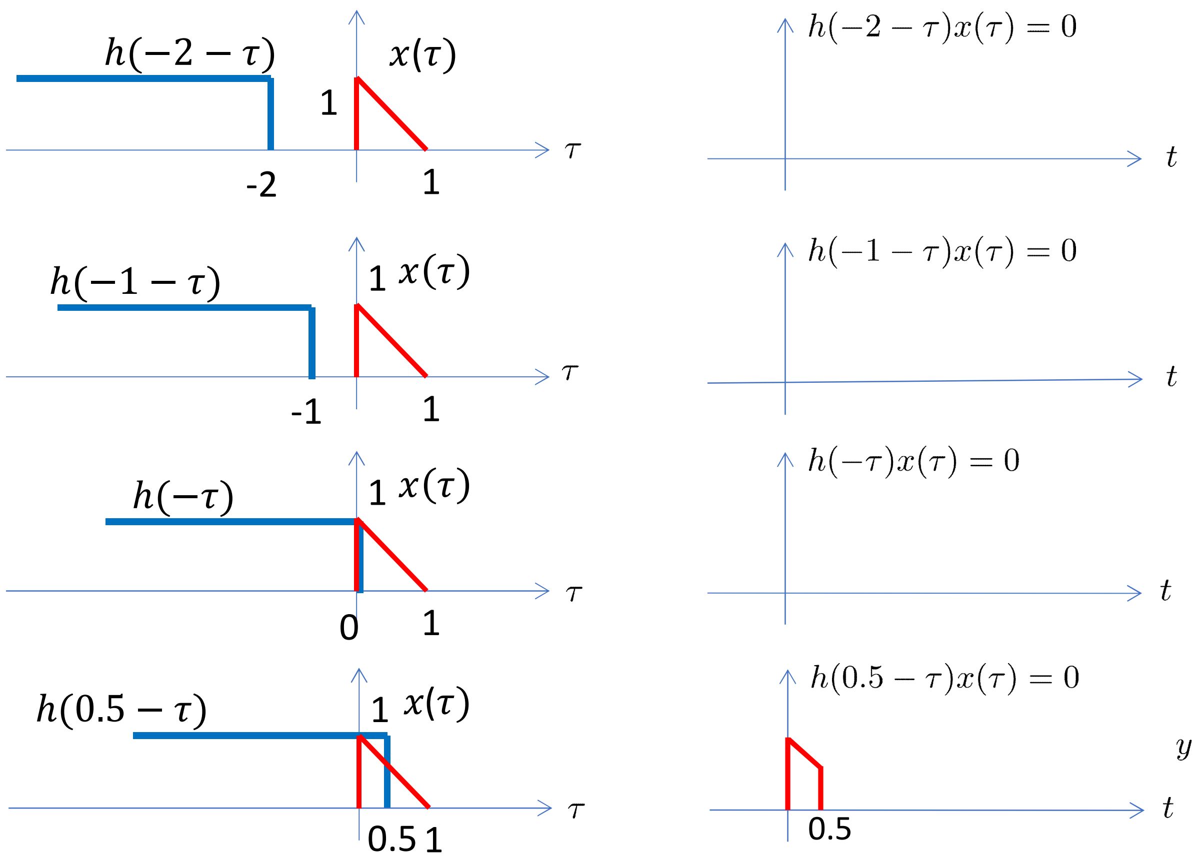
Convolution

Finding $x(\tau)h(t-\tau)$ for different values of t and its integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



$$y(0.5) = \int_0^{0.5} x(\tau)d\tau = 0.25 \times \frac{3}{2}$$

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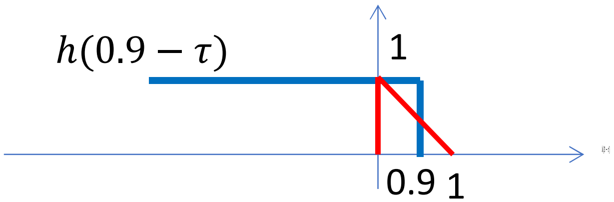
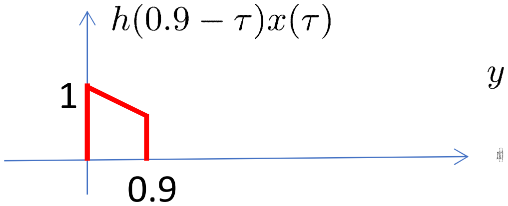
Convolution

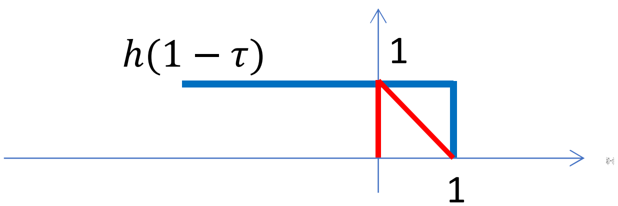
Finding $x(\tau)h(t-\tau)$ for different values of t and its integral (cont.):

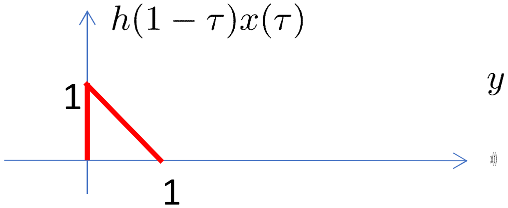
$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$



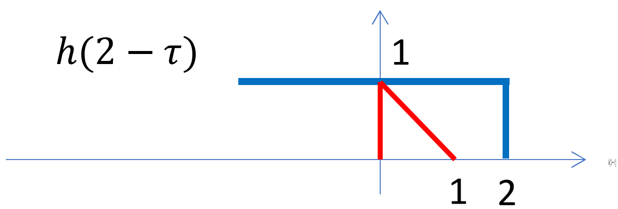


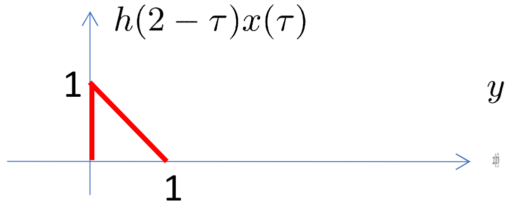
$$y(0.9)=\int_0^{0.9}x(\tau)d\tau=0.5\times\frac{-0.01}{2}$$





$$y(1)=\int_0^{0.9}x(\tau)d\tau=0.5$$





$$y(2)=\int_0^{2}x(\tau)d\tau=0.5$$

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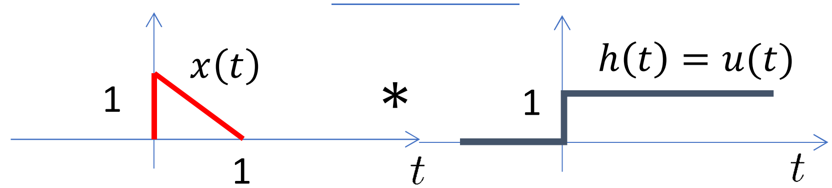
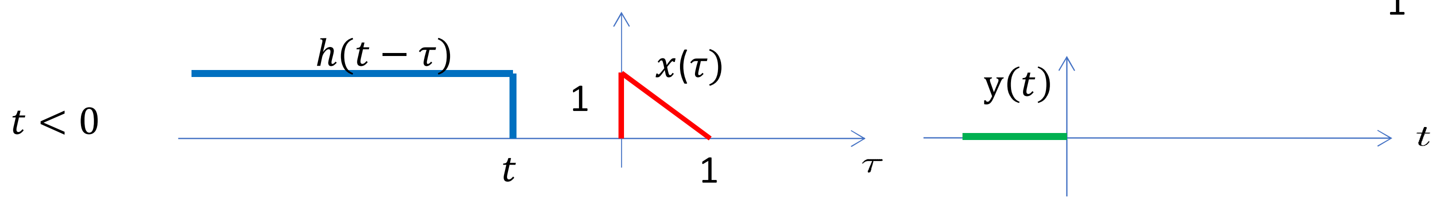
Convolution

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

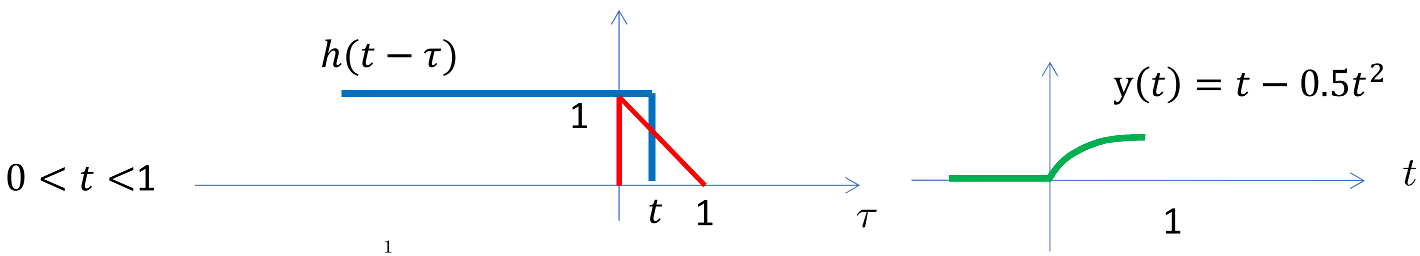


Generalizing for all t:

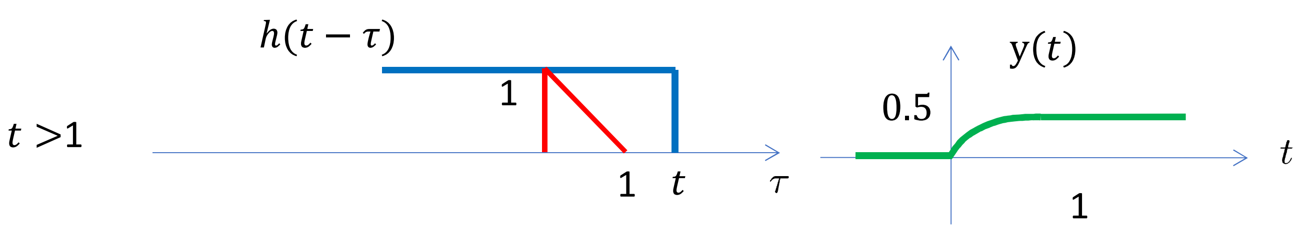
1- For $t<0$ there is no overlap between $x(\tau)$ and $h(t-\tau)$Therefore $x(\tau)\times h(t-\tau)=0$ and $y(t)=0$.



2- For $0<t<1$ there is a partial overlap: $y(t)=\int_0^t x(\tau)d\tau=\int_0^t(-\tau+1)d\tau=(-\frac{\tau^2}{2}+\tau)|_0^t=t-\frac{t^2}{2}$



3- For $t>1$ there is a full overlap: $y(t)=\int_0^1 x(\tau)d\tau=0.5$



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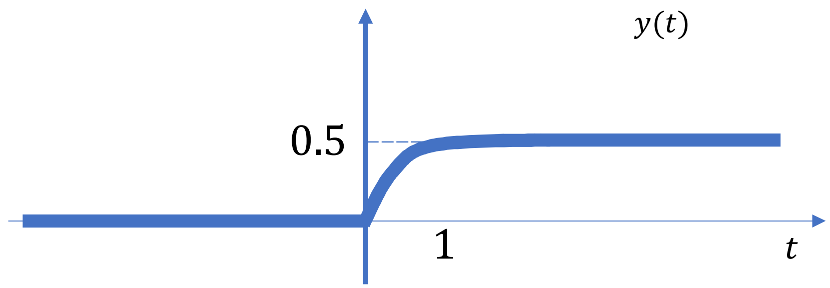
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Convolution

Plot $y(t)$ as function of t



To check whether the function is concave or convex between 0 and 1 you can try finding the value at $t = .5$. Here this value is $3/8$ which is larger than $0.5/2$ and makes the function concave.

Now try the same problem with flipping $x(t)$, try to find the following integral:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

The same wil be the same as what we have found previously.

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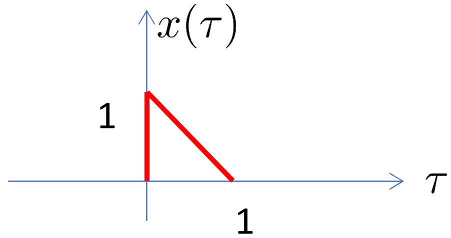
Convolution

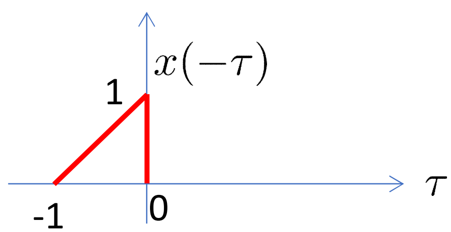
First build $x(t-\tau)$

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

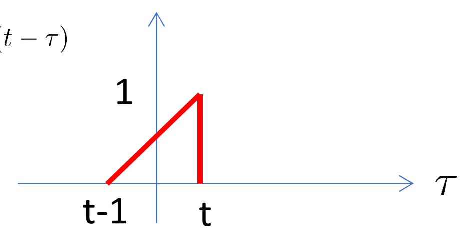
Step 1: Draw $x(\tau)$ by changing the independent variable from t to τ



Step 2: Flip $x(\tau)$ to show $x(-\tau)$



3- Move the value $x(-\tau)$ at zero to t to built $x(t-\tau)$



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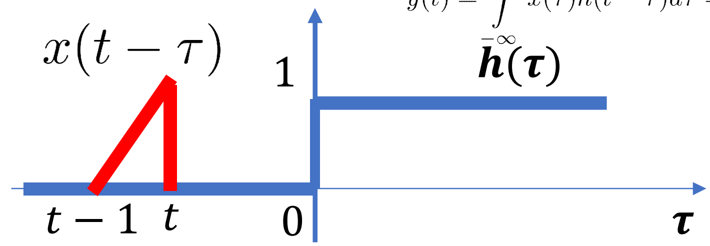
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Convolution

1- For $t < 0$ there is no overlap between $h(\tau)$ and $x(t-\tau)$:

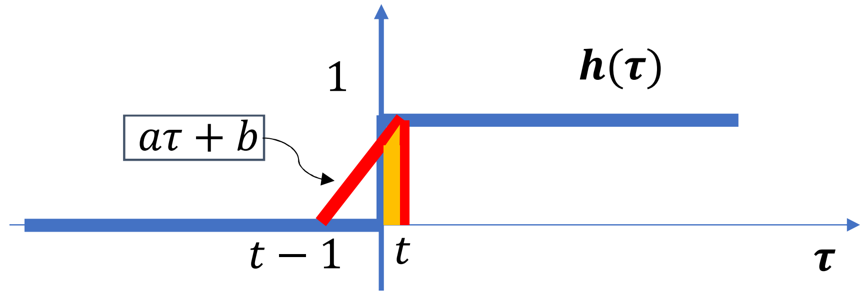
Therefore $y(t) = 0$



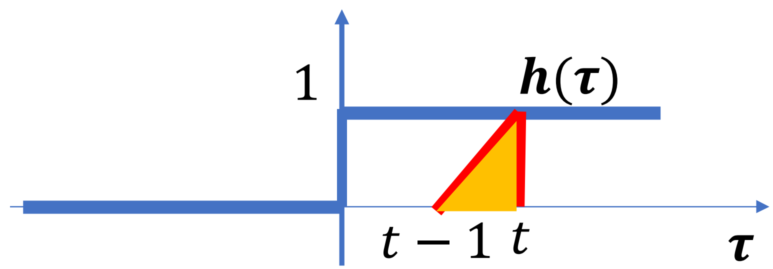
2- For $0, t < 1$ there is a partial overlap between $h(\tau)$ and $x(t-\tau)$:

$$\begin{cases} at+b=1 \\ a(t-1)+b=0 \end{cases} \rightarrow \begin{cases} a=1 \\ b=-t+1 \end{cases} \rightarrow \text{The line equation:} \boxed{\tau+1-t}$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{0}^{t} (\tau - t + 1)d\tau = [\tau^2/2 + (1-t)\tau]|_0^t = t - t^2/2$$



3- For $t > 1$ there is a full overlap: $y(t) = \int_0^1 x(\tau)d\tau = 0.5$



So the final answer is identical to what we had calculated before

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

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$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

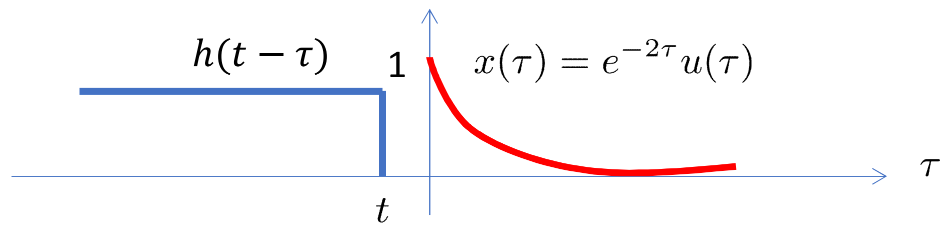
Convolution

Example 2: Find the output of the system with $h(t) = u(t)$ for input $x(t) = e^{-2t}u(t)$

Solution:

First plot $h(t-\tau)$ through the three steps.

1- For $t < 0$ there is no overlap, therefore $y(t) = 0$.

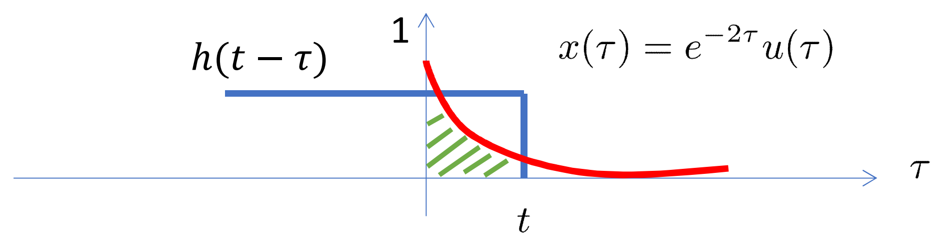


2- For $t > 0$ there is a partial overlap, and the convolution is equal to the area under the overlapped section.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{0}^{t} e^{-2\tau}d\tau$$

$$= \left.\frac{e^{-2\tau}}{-2}\right|_0^t = \frac{1}{2} - \frac{e^{-2t}}{2}$$



So the final answer is

$$\boxed{y(t) = \left(\frac{1}{2} - \frac{e^{-2t}}{2}\right)u(t)}$$

Try flipping and shifting $x(t)$ and verify that you get the same answer.

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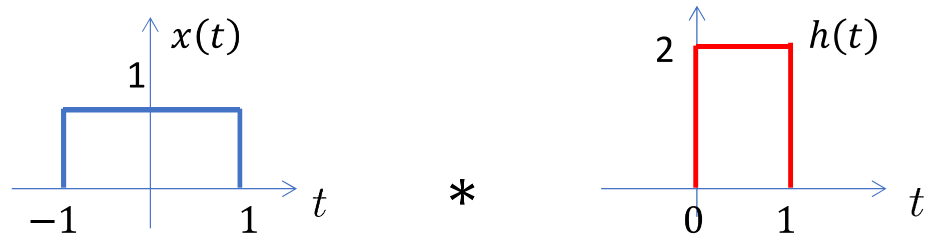
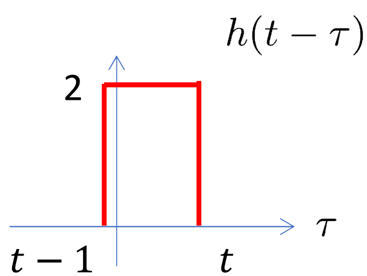
Convolution

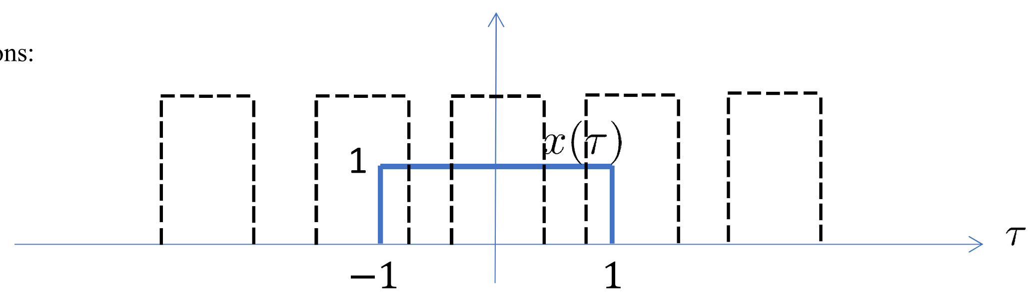
Example 3: Find the output $y(t)$ for the following $h(t)$ and input $x(t)$. $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$



First build $h(t-\tau)$



Based on value of t we have different overlapping sections:



Find the different ranges of t

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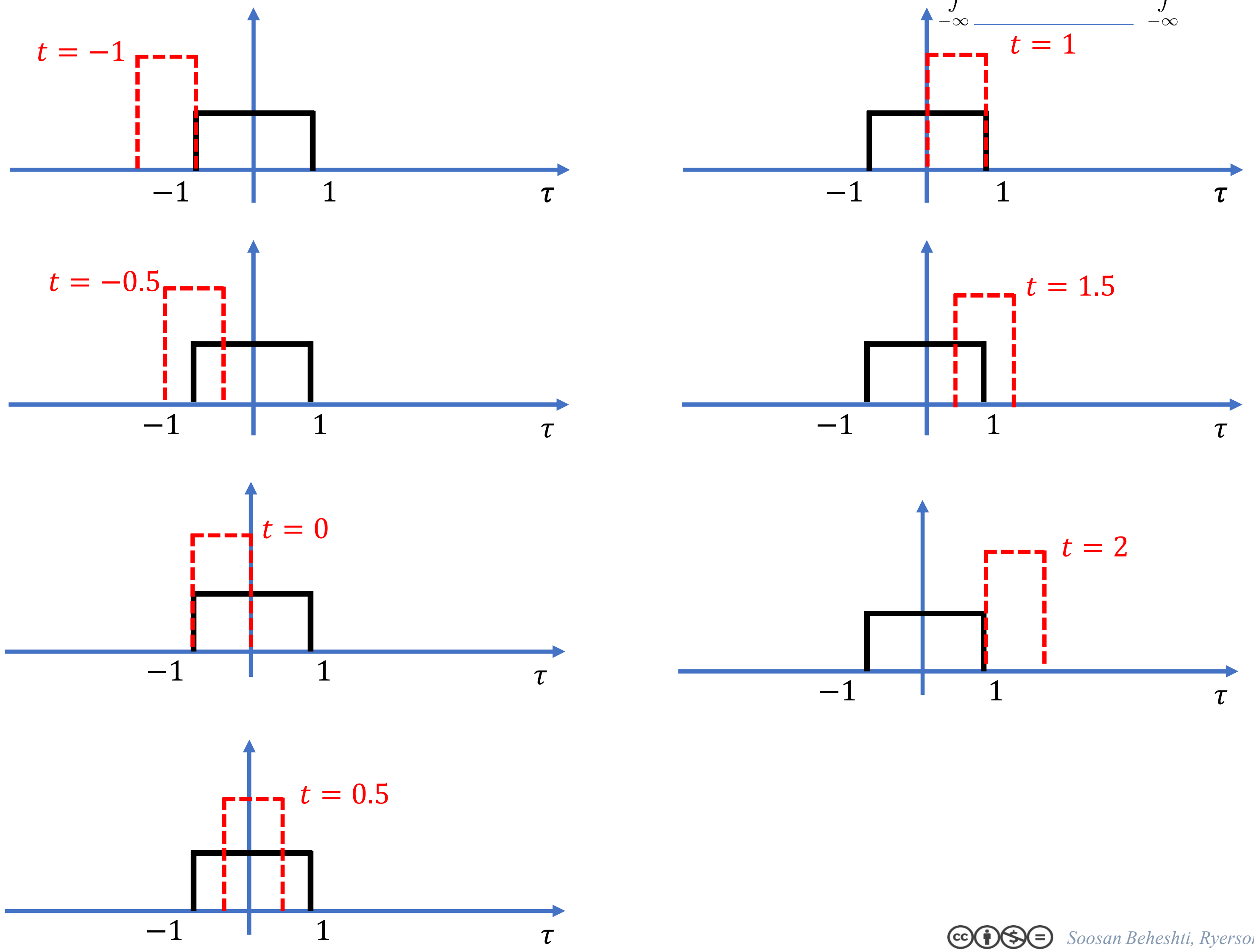
Convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



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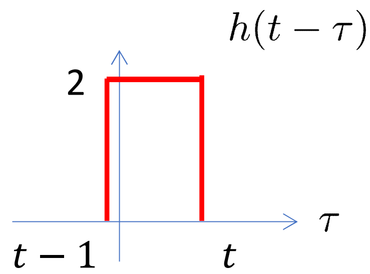
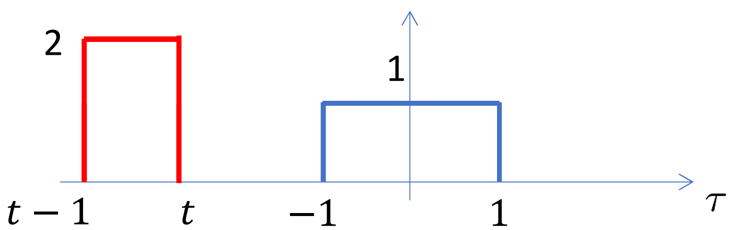
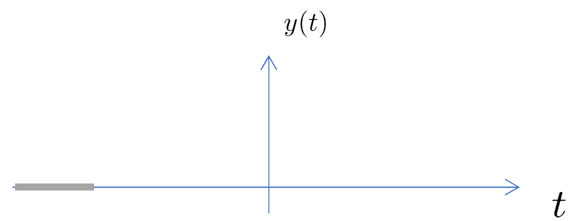
Convolution



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

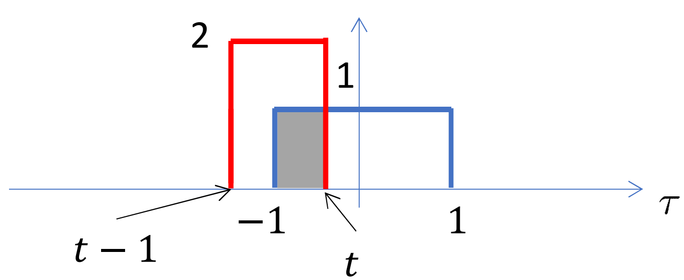
1- For $t < -1$ there is no overlap and $y(t) = 0$.

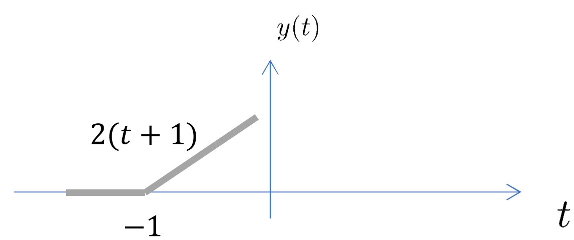




2- For $-1 < t < 0$ there is a partial overlap

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-1}^{t} 2d\tau = 2(t+1)$$





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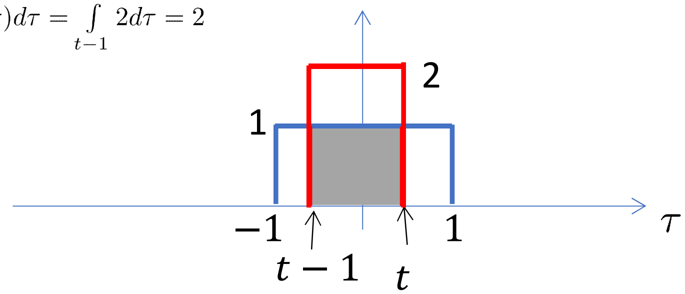
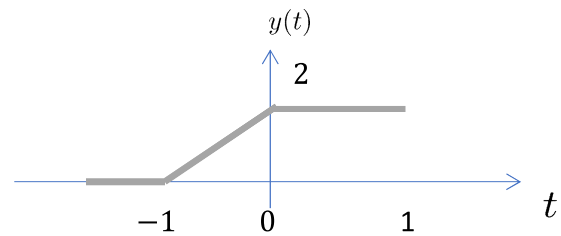
Convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

3- For $0 < t < 1$ we have a full overlap

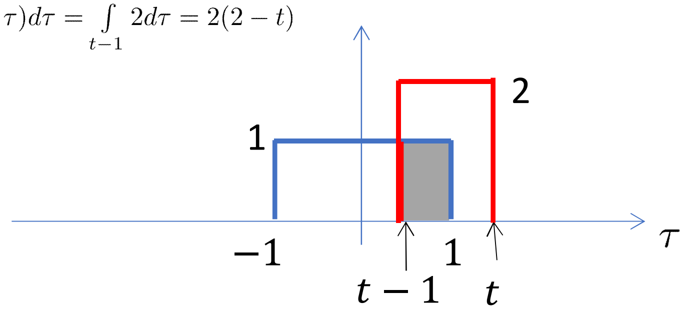
$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{t} 2d\tau = 2$

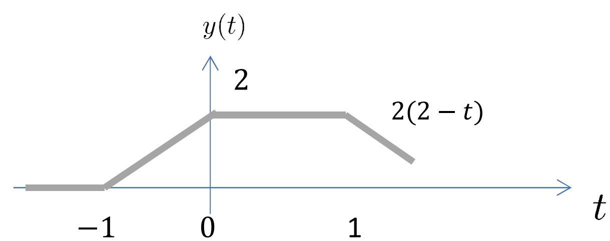




4- For $0 < t-1 < 1 \rightarrow 1 < t < 2$ there is a partial overlap:

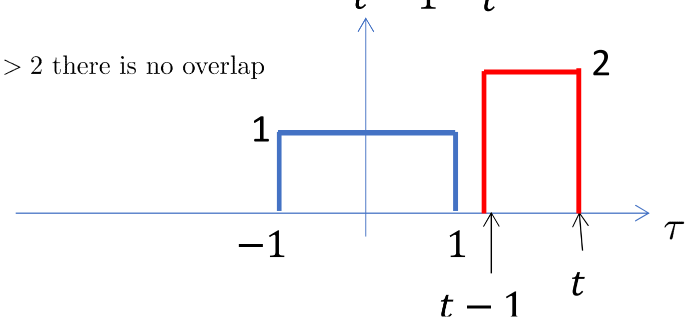
$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{1} 2d\tau = 2(2-t)$

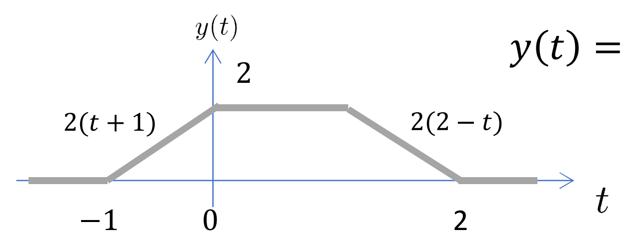




5- For $t-1 > 1 \rightarrow t > 2$ there is no overlap

$y(t) = 0$.





$$y(t) = x(t) * h(t)$$

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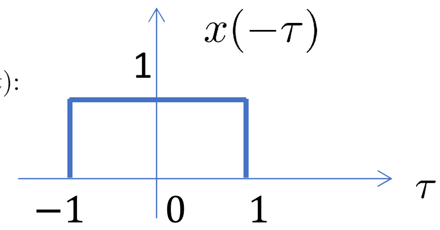
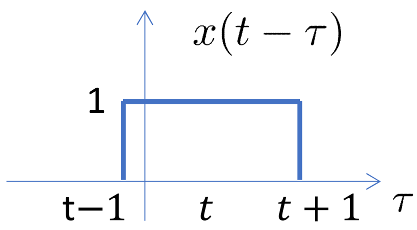
Convolution

Try the same problem, this time by flipping and shifting $x(t)$:

$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$





Note: Width of convolution of two finite width signals with widths of T_1 and T_2 is always $T_1 + T_2$

In addition if one signal starts at t_1 and the other signal starts at t_2, the covolution of two signals starts at $t_1 + t_2$.

Consequently, if the two signals end at t_3 and t_4 the convolution ends at $t_3 + t_4$.

Check for this example!

...and for all convolutions you solve

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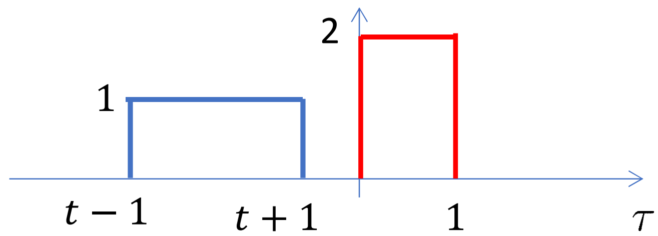
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

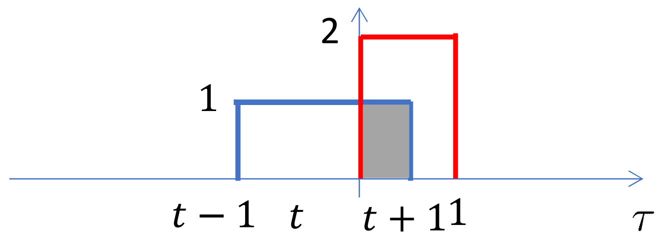
Convolution

Sliding $x(t-\tau)$ over $h(t)$:

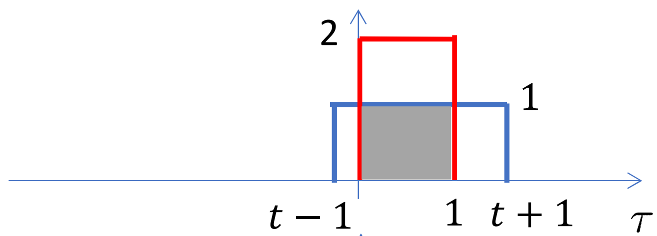
1- For $t+1<0 \rightarrow t<-1$ there is no overlap, therefore, $y(t)=0$.



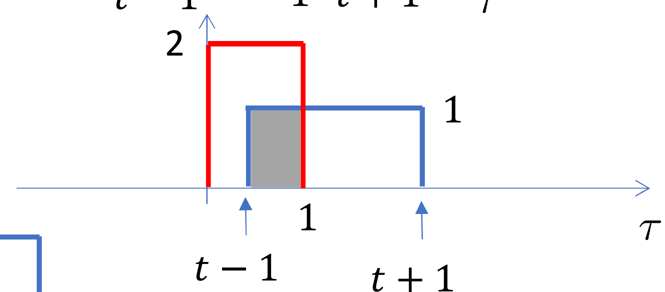
2- For $0<t+1<1 \rightarrow -1<t<0$ there is partial overlap



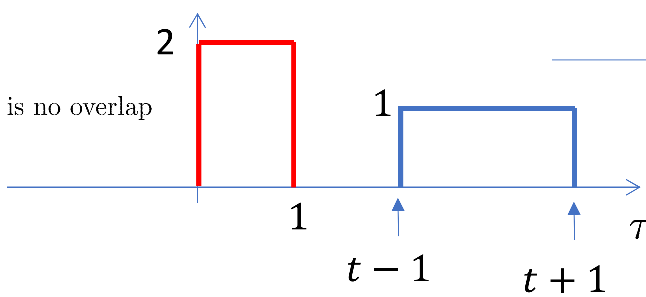
3- For $t+1>1$ and $t-1<0 \rightarrow 0<t<1$ there is a full overlap



4- For $0<t-1<1 \rightarrow 1<t<2$ there is a partial overlap



5- For $t-1>1 \rightarrow t>2$ there is no overlap



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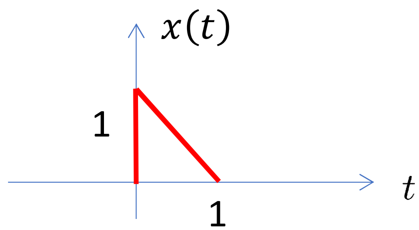
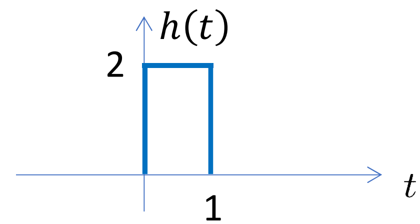
Convolution

Example 4: Find the output of previous system $h(t)$ to the following input $x(t)$:

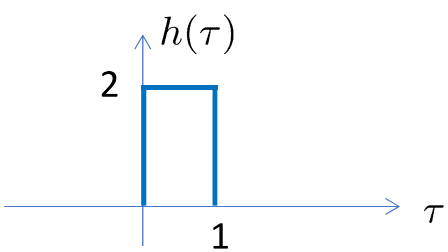




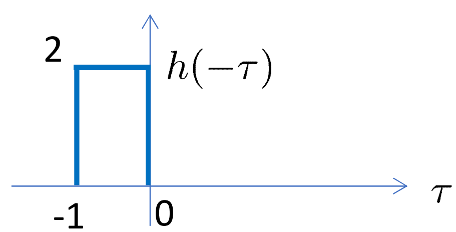
It is up to you to flip and shift $x(t)$ or $h(t)$. Start with the one that is easier!
For example here we start with $h(t)$ as it seems to be easier.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

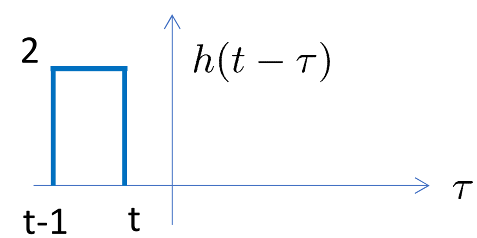
Step 1: Draw $h(\tau)$ by changing the independent variable from t to τ



Step 2: Flip $h(\tau)$ to show $h(-\tau)$



3- Move the value $h(-\tau)$ at zero to t to built $h(t-\tau)$



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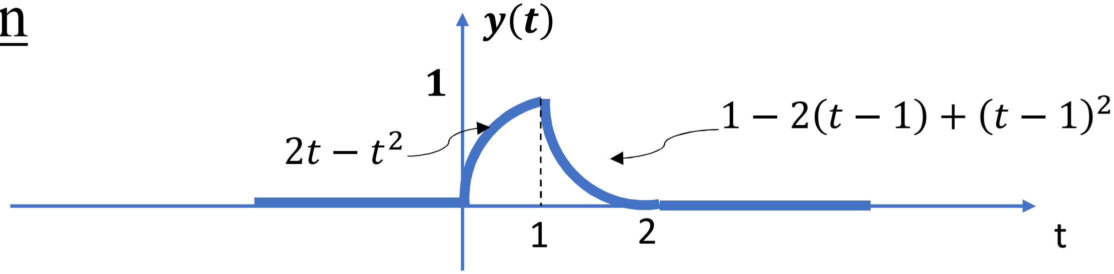
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

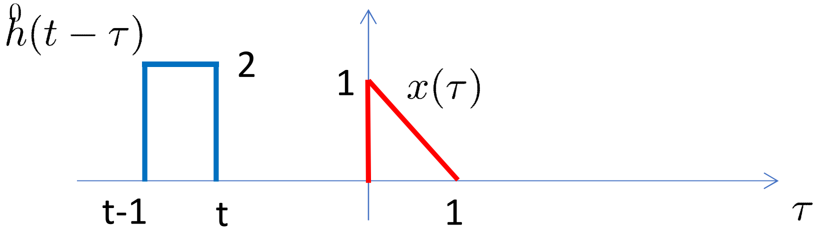
Convolution

Final answer:



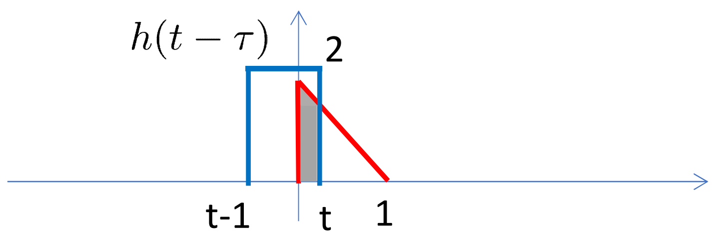
Solution:

1- For $t < 0$ there is no overlap and $y(t) = 0$



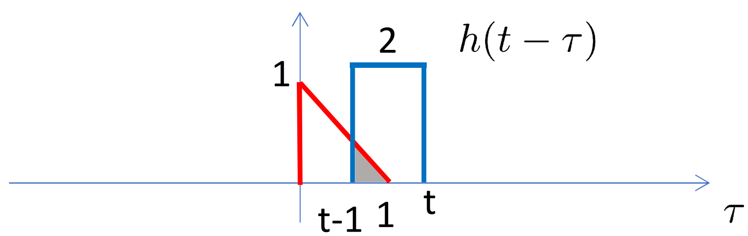
2- For $0 < t < 1$ there is a partial overlap and

$$y(t) = \int_{0}^{t} 2(-\tau + 1)d\tau = 2t - t^2$$

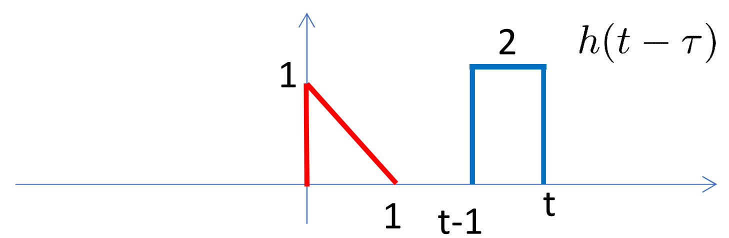


3- For $1 < t < 2$ there is a partial overlap and

$$y(t) = \int_{t-1}^{1} 2(-\tau + 1)d\tau = 1 - 2(t-1) + (t-1)^2$$



4- For $t > 2$ there is no overlap and $y(t) = 0$



Try flipping and shifting $x(t)$ and verify that you get the same answer.

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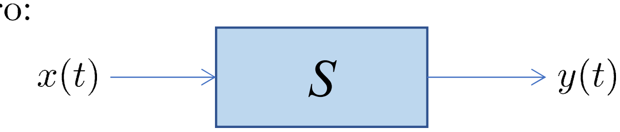
Initial Condition and LTIDE systems

$$\left(D^N + a_1 D^{N-1} + \ldots + a_N\right) y(t) = \left(b_{N-M} D^M + b_{N-M-1} D^{M-1} + \ldots + b_N\right) x(t)$$

The convolution answer is for casual LTI system **at initial rest** or **zero state (ZS)** system which has input $x(t)$ and impulse response $h(t)$. In this case it is assumed that $y(0)$ and derivatives of $y(t)$ up to order $N-1$ at zero are zero:



$$y(0) = 0, y^{(1)}(0) = 0, \cdots, y^{(N-1)}(0) = 0$$

The output of the system in this case is denoted as y_{ZS} where ZS is for zero state:

$$y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

Note that this system is LTI in zero state status. But in presence of initial conditions we have to be careful with using linearity and time invariance properties. Why?

In presence of initial condition we have at least one nonzero values for $y(0)$ and its derivatives up to $N-1$th order. Find the output of the system for its initial conditions separatelty. This output is denoted by zero input response y_{zir}. First set the characteristic mode $C(t)$ to the values provided for $y(0), y^{(1)}(0), \cdots, y^{(N-1)}(0)$ at zero to find the coeffitients, i.e, $y(0) = C(0)$, $y^{(1)}(0) = C'(0), \cdots$. Using those coefficients, the system response after $t = 0$ is:

$$y_{zir}(t) = C(t)u(t)$$

For example for the case of non repeated roots Characteristic mode is $C(t) = (c_1 e^{\lambda_1 t} + \ldots + c_N e^{\lambda_N t})$

The finial answer is

$$y(t) = y_{zs}(t) + y_{zir}(t)$$