Signals and Systems I

Lecture 6

Last Lecture

- Impulse Response of LTI Systems
- Convolution

$$y(t) = x(t) * y(t) = y(t) * x(t)$$
$$y(t) = \int x(\tau)h(t-\tau)d\tau = \int h(\tau)x(t-\tau)d\tau$$

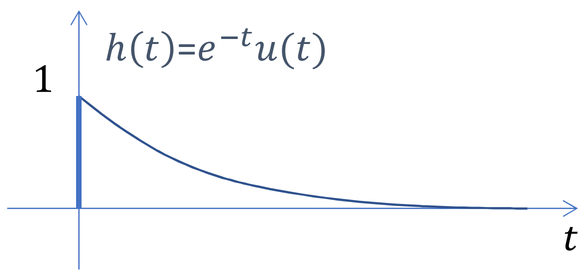
Convolution is commutative!

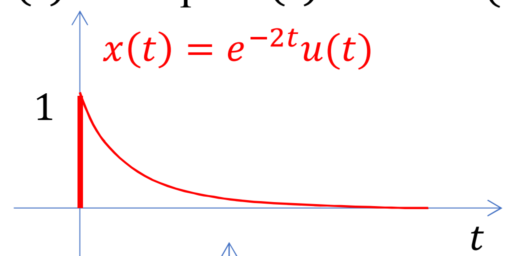
Today

- More on Convolution
- Convolution of $\delta(t)$ and its shifted version
- LTI System Stability & Causality
- Convolution Properties
- LTI System interconnections
- Introduction to Fourier Series (If we have time)

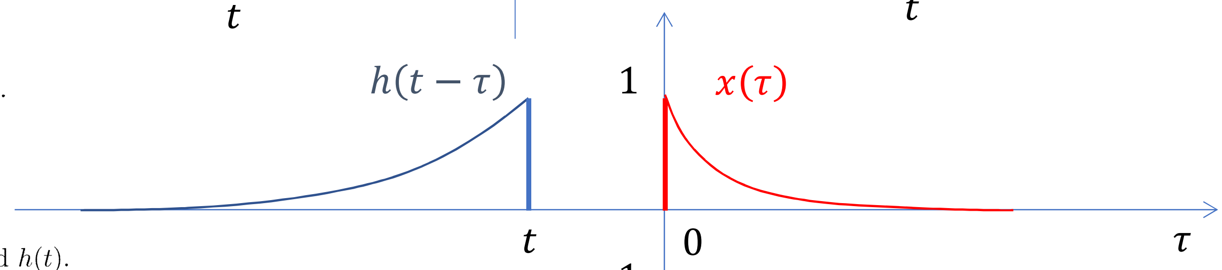
Convolution

Example: Find the output for the given system $h(t) = e^{-t}u(t)$ and input $x(t) = e^{-2t}u(t)$

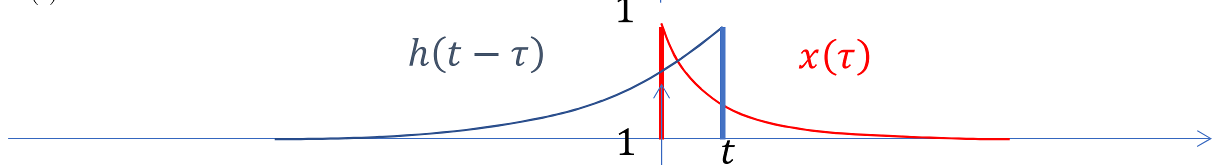




1- For $t < 0$ there is no overlap and $y(t) = 0$.



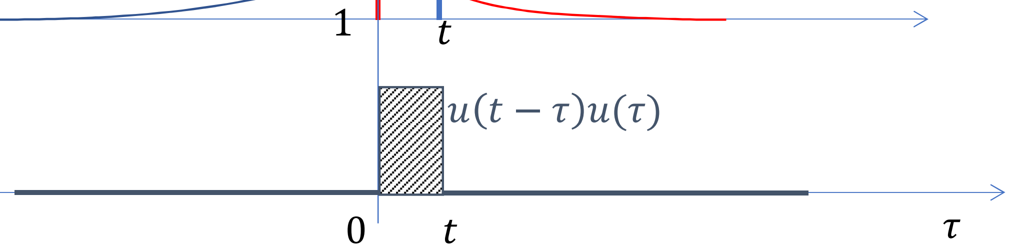
2- For $t > 0$ there is overlap between $x(t)$ and $h(t)$.



$$x(\tau)h(t-\tau) = e^{-\tau}u(\tau).e^{-2(t-\tau)}u(t-\tau)$$
$$= e^{-\tau}e^{-2(t-\tau)}u(\tau)u(t-\tau)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$= \int_0^t e^{-\tau}e^{-2(t-\tau)} = e^{-2t}\int_0^t e^{-\tau}e^{2\tau} = e^{-2t}\int_0^t e^{\tau} = e^{-2t}e^{\tau}\Big|_0^t$$
$$= e^{-2t}(e^t - 1) = (e^{-t} - e^{-2t}) \quad \text{for} \quad t > 0$$
$$= (e^{-t} - e^{-2t})u(t)$$

Plot $y(t)$



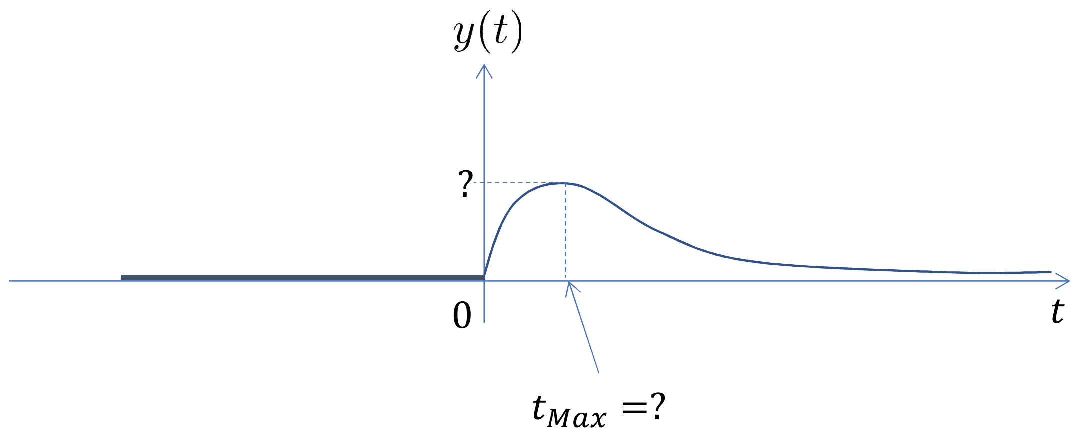
Before taking care of the function inside the integral, find the boundaries of te integral.

Convolution

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

1- Check for $t = 0, t = \infty$.
$y(0) = 0, \quad y(\infty) = \lim_{t\to\infty} (e^{-t} - e^{-2t}) = 0$

Also, $e^{-t} > e^{-2t}$ for $t > 0 \Rightarrow e^{-t} - e^{-2t} > 0$
So this is a positive function.



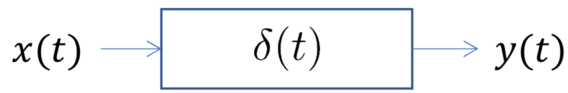
2- Find t_{Max}:

$$t > 0, \quad \frac{\partial}{\partial t} y(t) = 0 \Rightarrow -e^{-t} + e^{-2t} = 0 \Rightarrow 2e^{-t} = 1$$

$$t_{max} = ln2 \rightarrow y(t_{max}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Identity System

What is the output of system with $h(t) = \delta(t)$



$= x(t)$
this is the identity system

$$y(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Where t is a number and τ is the random variable.

$$= \int_{-\infty}^{\infty} x(\tau) \overbrace{\delta(t-\tau)}^{\text{This term is only nonzero at } \tau=t} d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau$$

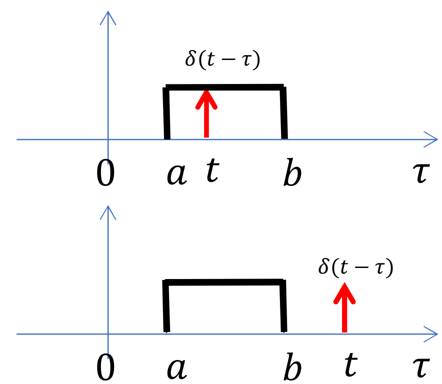
$$= x(t) \underbrace{\int_{-\infty}^{\infty} \delta(t-\tau)d\tau}_{1}$$

$$= x(t)$$

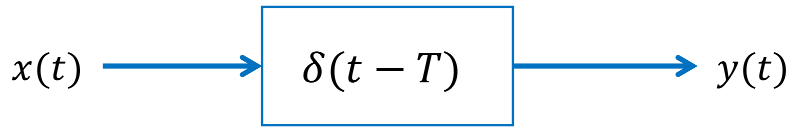
Reminder:

$$\int_a^b \delta(t-\tau)d\tau = 1 \quad \text{if} \quad a < t < b \text{ or } t \in [a,b]$$

$$\int_a^b \delta(t-\tau)d\tau = 0 \quad \text{if} \quad t \notin [a,b]$$



Pure Delay System

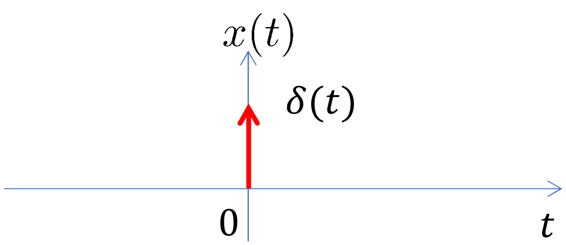


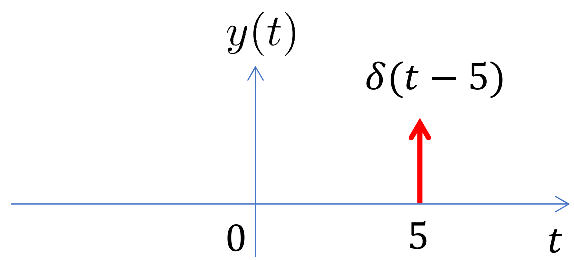
$= x(t-T)$

this is the pure delay system

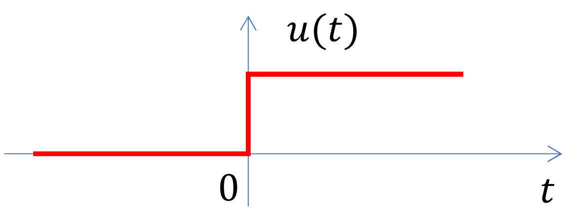
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-T-\tau)d\tau = x(t-T)$$

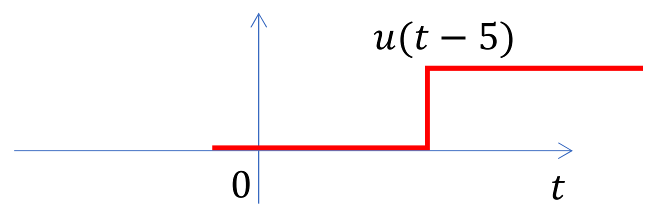
Example with $T = 5$





$$h(t) = \delta(t-5)$$

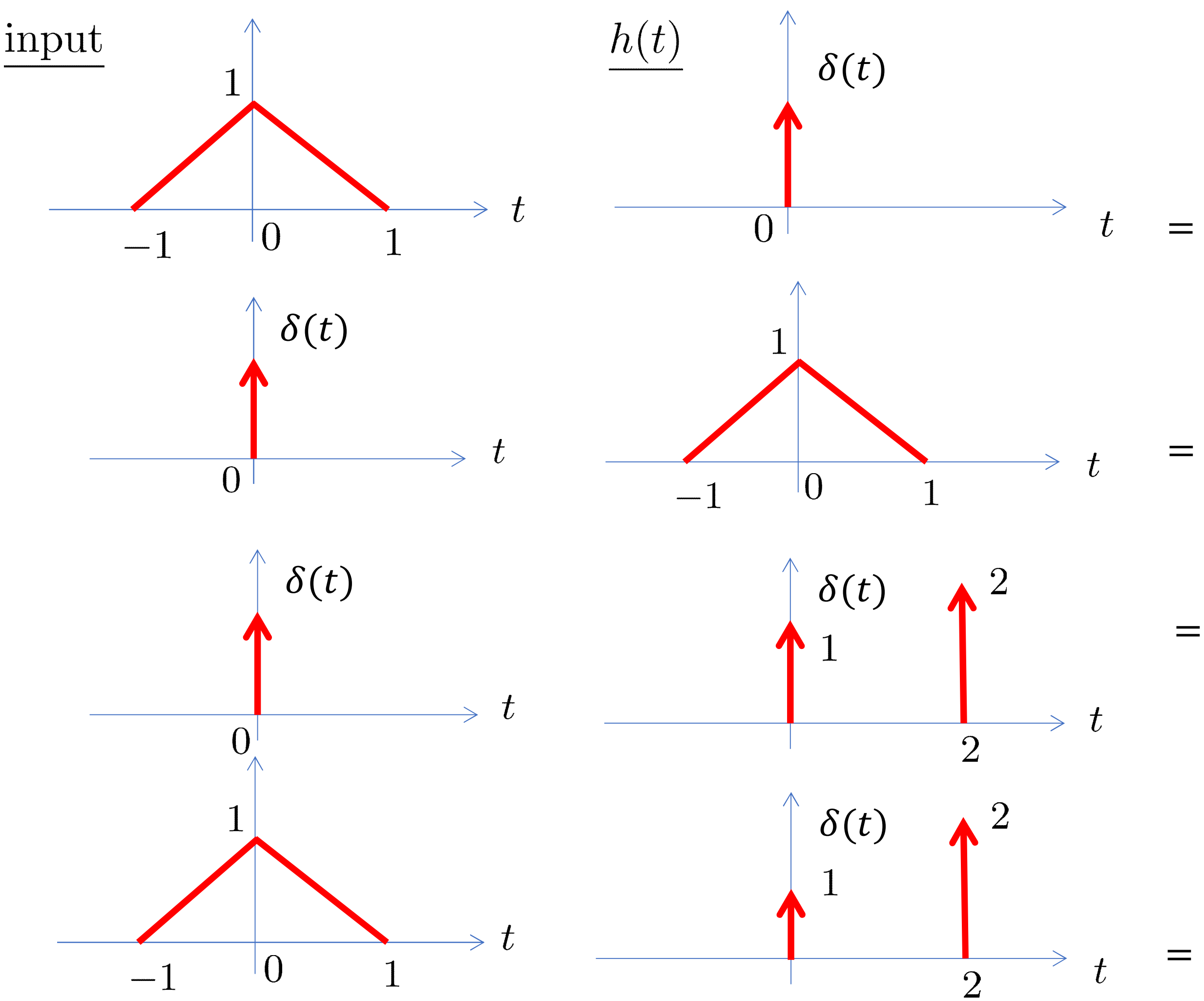




Convolution with $\delta(t)$

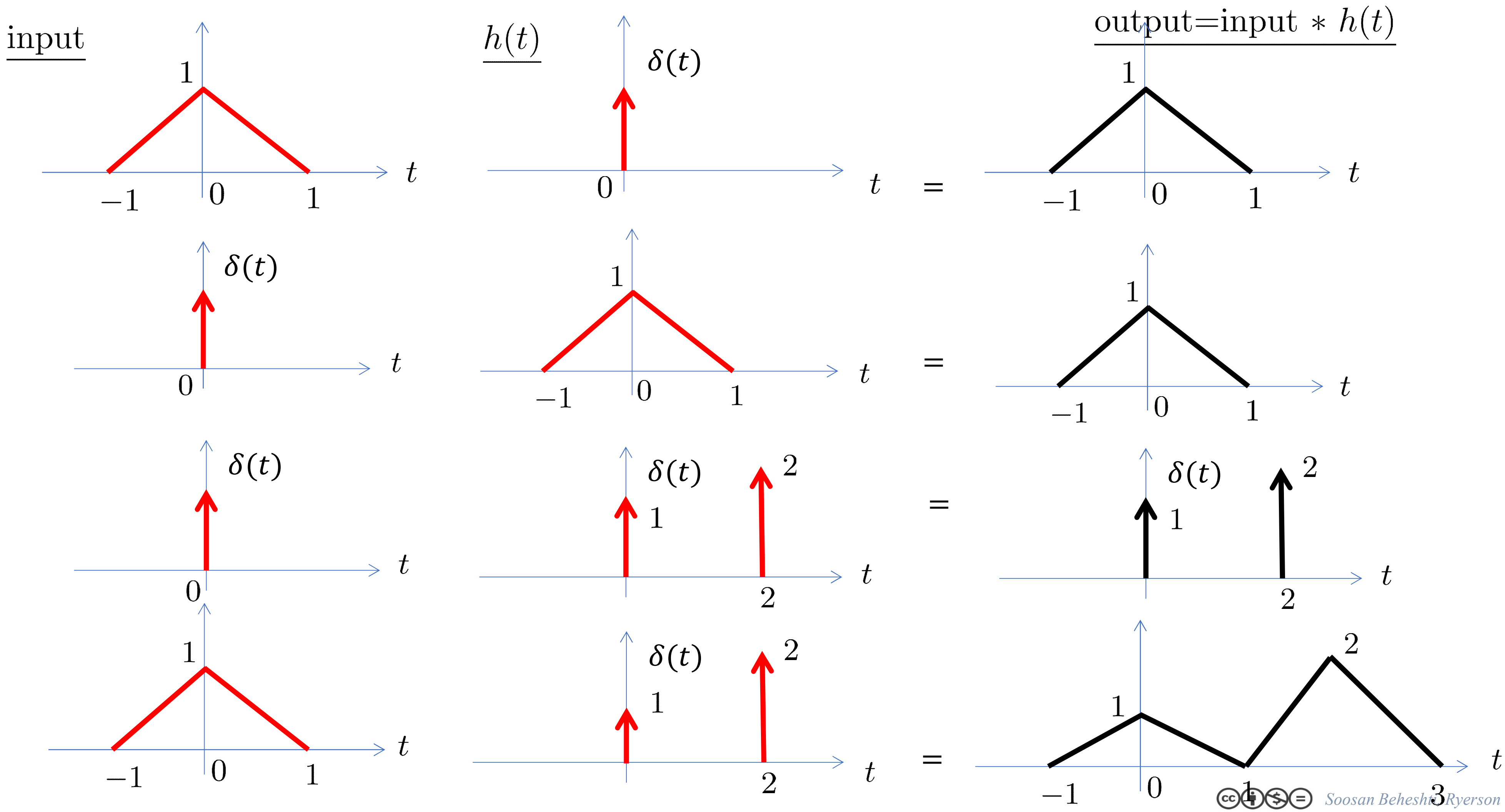
Find the outputs:

$$\text{output} = \text{input} * h(t)$$

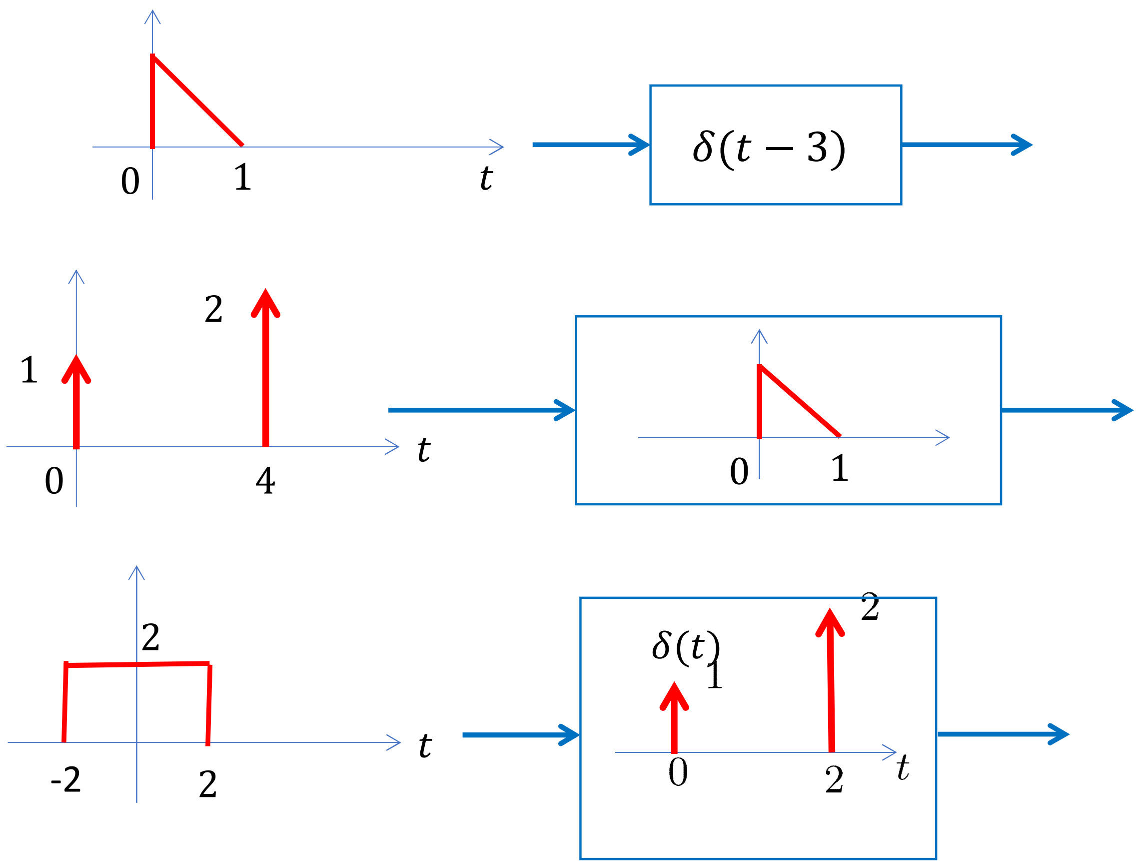


Convolution with $\delta(t)$

Find the outputs:

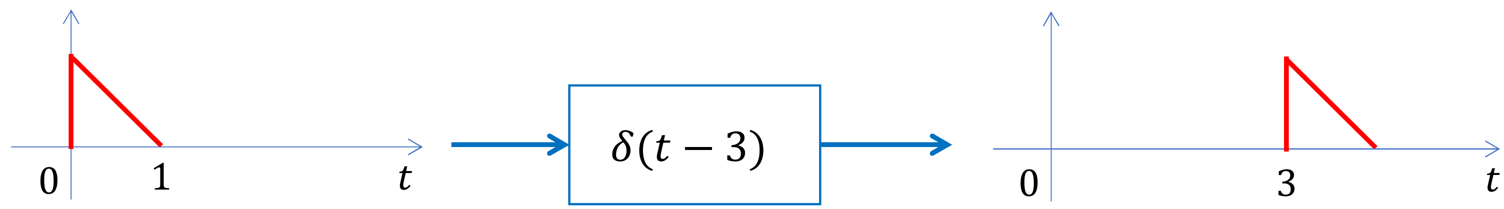


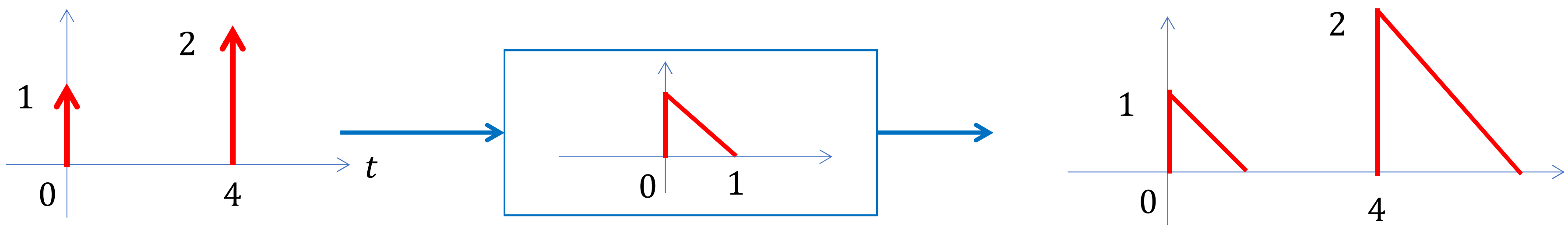
Convolution with $\delta(t)$



Convolution with $\delta(t)$

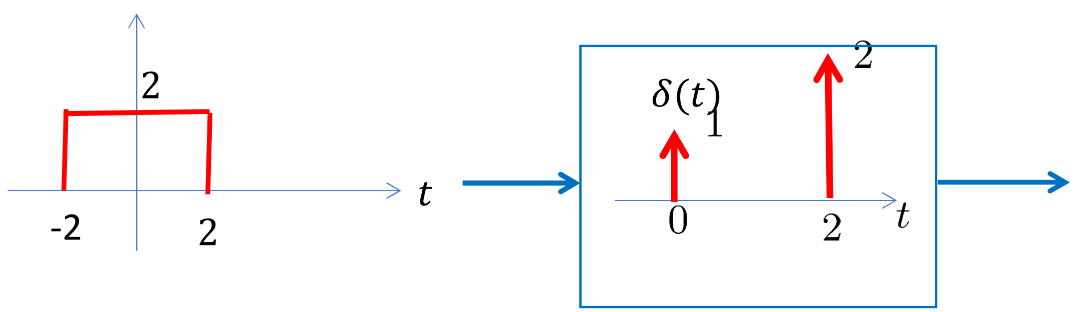
$y(t) = x(t-3)$



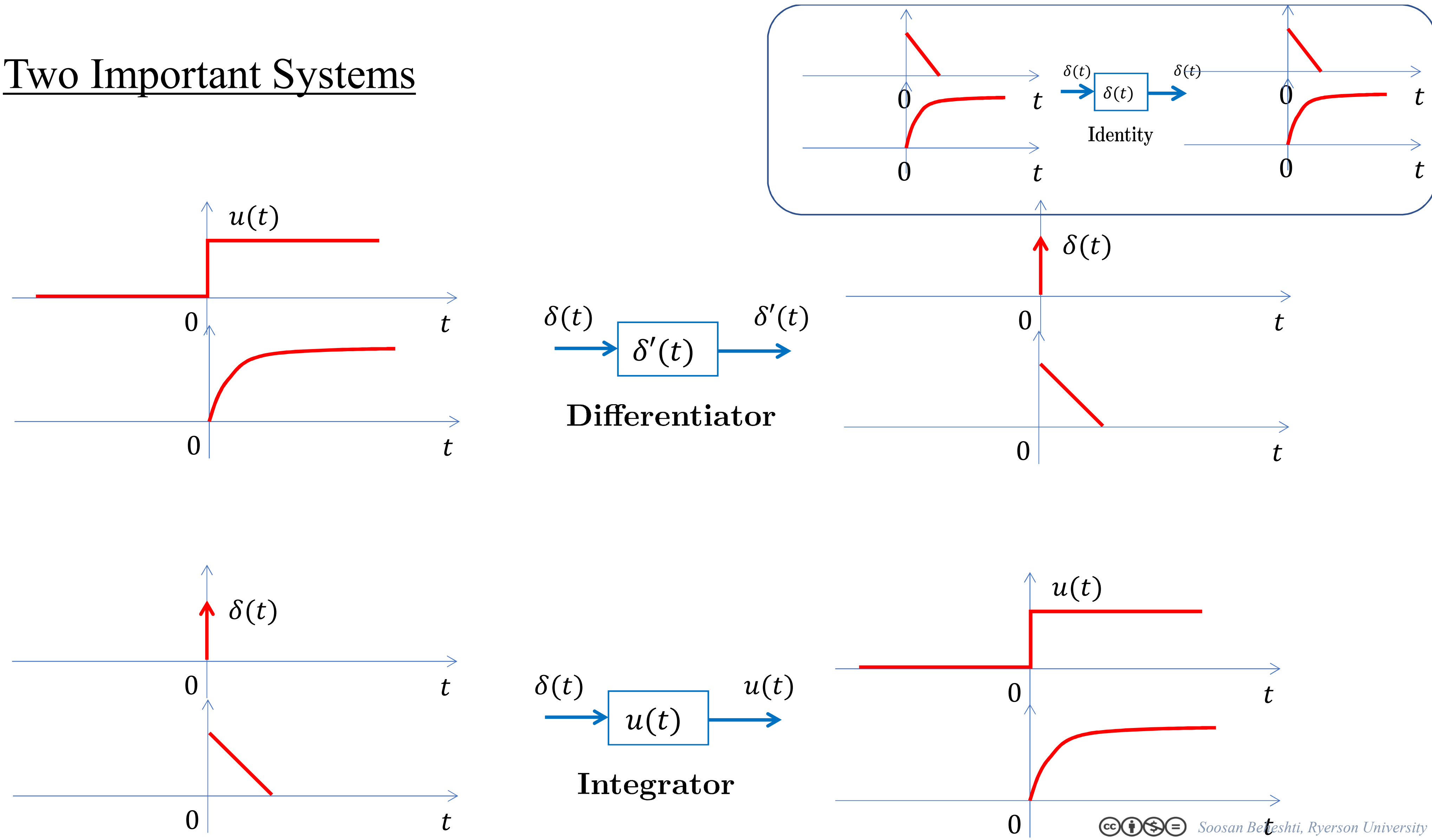


$x(t) = \delta(t) + 2\delta(t-4)$

$$y(t) = x(t) * h(t) = (\delta(t) + 2\delta(t-4)) * h(t)$$
$$= \delta(t) * h(t) + 2\delta(t-4) * h(t) = h(t) + 2h(t-4)$$



Two Important Systems



Useful Convolution formulas for $t^n u(t)$

$$u(t) * \delta(t) = u(t)$$

$$u(t-a) * \delta(t-b) = u(t-a-b)$$

$$u(t) * u(t) = tu(t)$$

$$u(t-a) * u(t-b) = (t-a-b)u(t-a-b)$$

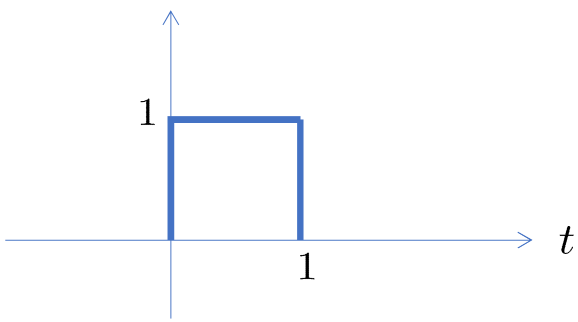
$$u(t) * tu(t) = \frac{t^2}{2}u(t)$$

$$u(t-a) * (t-b)u(t-b) = \frac{(t-a-b)^2}{2}u(t-a-b)$$

$$u(t) * t^n u(t) = \frac{t^{n+1}}{n+1}u(t)$$

$$u(t-a) * (t-b)^n u(t-b) = \frac{(t-a-b)^{n+1}}{n+1}u(t-a-b)$$

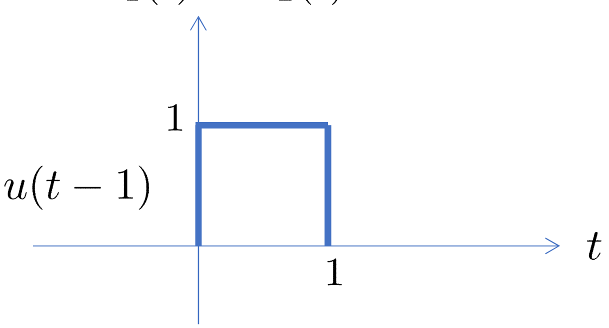
$h_1(t) = h_2(t)$



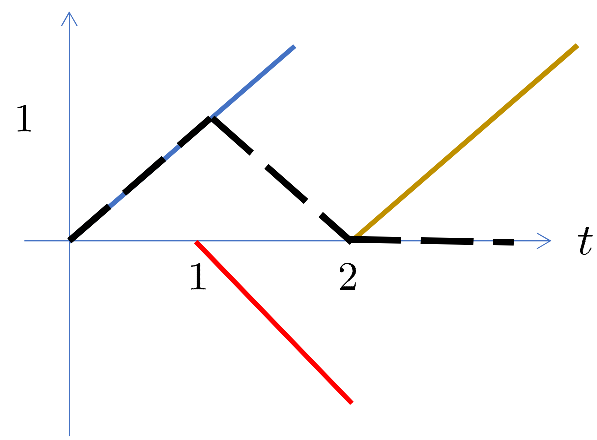
Use the above to find $h(t) = h_1(t) * h_2(t)$

Useful Convolution formulas for $t^n u(t)$

Example: Find the convolution of $h_1(t) * h_2(t)$ where $h_1(t) = h_2(t) = u(t) - u(t-1)$

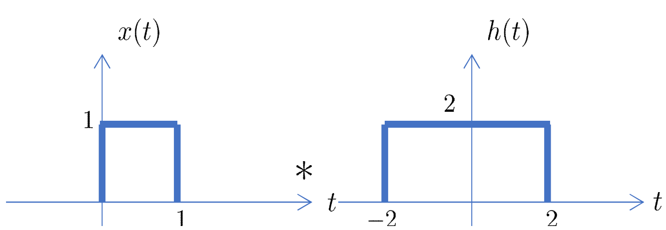


$$
\begin{aligned}
h_1(t) * h_2(t) &= (u(t) - u(t-1)) * (u(t) - u(t-1)) \\
&= u(t) * u(t) - u(t) * u(t-1) - u(t-1) * u(t) + u(t-1) * u(t-1) \\
&= tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2) \\
&= \begin{cases} t & 0 < t < 1 \\ -t+2 & 1 < t < 2 \\ 0 & t > 2 \end{cases}
\end{aligned}
$$



$$h_1(t) * h_2(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$$

Find convolution of these two signals using flip & shift and also this method:



LTI Systems: Stability Test

An LTI system is Bounded-input/Bounded-output (BIBO) stable, if & only if, impulse response of the system $h(t)$ is absolutely integrable: $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

Why? Assume that input is bounded: $|x(t)| < C$, for all t then we have:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$|y(t)| \leq \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq C \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

So if $\int_{-\infty}^{\infty} |h(\tau)| d\tau$ is bounded, then $|y(t)|$ is also bounded.

On the other hand if $\int_{-\infty}^{\infty} |h(\tau)| d\tau = \infty$ then consider the following bounded input ($|x(t)| = 1$):

$$x(t) = \begin{cases} +1 & \text{if} \quad h(t) > 0 \\ -1 & \text{if} \quad h(t) < 0 \end{cases}$$

Then we have

$$y(0) = \int_{-\infty}^{\infty} h(-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} |h(\tau)|d\tau = \infty$$

so the system is not stable!

LTI Systems: Stability Test

BIBO stability is the same as external stability. Internal (Asympototic) stability is when all the poles of the system are in left half plane which means that they have negative real part (thorough study of this stability is in Control courses that introduce state space models)

If an LTI causal system is both observable and controllable (these terms are defined in state space modeling in future courses) then internal stability and BIBO (external) stability can be evaluated by the roots of the characteristic polynomial:

- If all the roots, λs, have negative real parts, the system is both BIBO stable and interally stable.

- If at least one root has a positive real part, the system is unstable (both BIBO and internally).

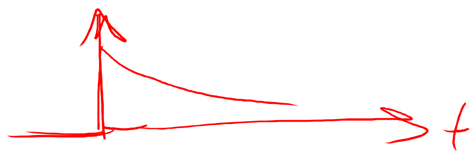
- If there are roots with zero real part and those roots are not repeated roots, the system is called marginally stable which is not BIBO stable.

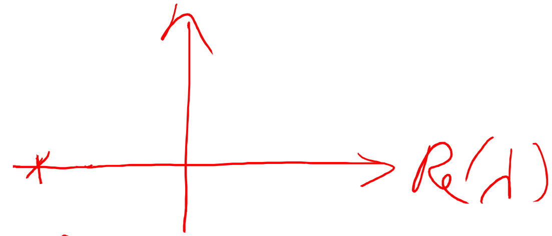
LTI Systems: Stability Test

$y'(t) + 5y(t) = x(t)$

$\lambda + 5 = 0 \rightarrow \lambda = -5$

$h(t) = e^{-5t} u(t)$





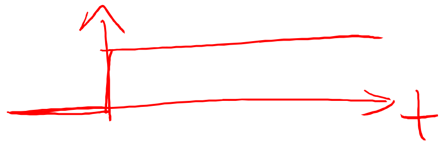
BIBO & Internally Stable

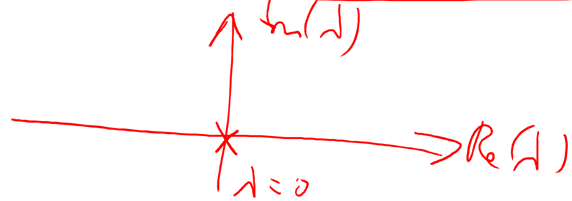
$\int |h(t)| dt < c$

$y'(t) = x(t)$

$\lambda = 0$

$h(t) = u(t)$





BIBO unstable

$\int |h(t)| dt = \infty$

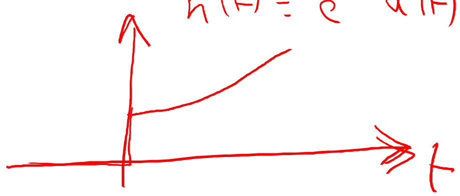
Marginally stable

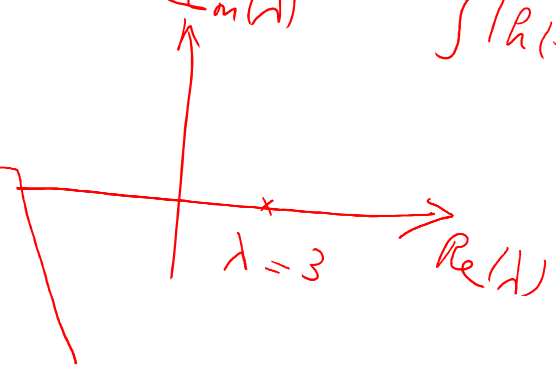
$y'(t) - 3y(t) = x(t)$

$\lambda - 3 = 0$

$\Rightarrow \lambda = 3$

$h(t) = e^{3t} u(t)$





BIBO unstable

$\int |h(t)| dt = \infty$

Internally unstable

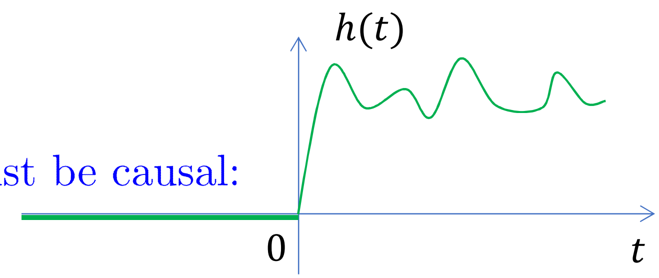
$y''(t) = x(t)$

?

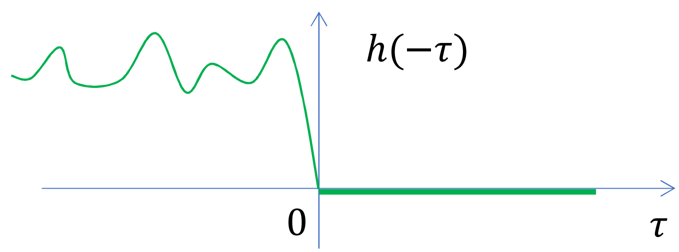
LTI Systems: Causality Test

An LTI system is Causal, if & only if, $h(t) = 0, \quad t \leq 0$. Why?

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

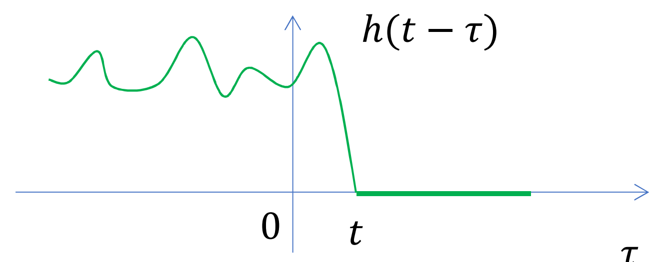


For LTI *system* to be causal, impulse response *signal* $h(t)$ must be causal:
$h(t) = 0, \quad t \leq 0$



Since $h(t-\tau)$ is zero after t, therefore:

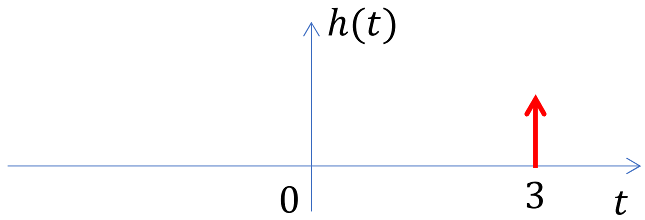
$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

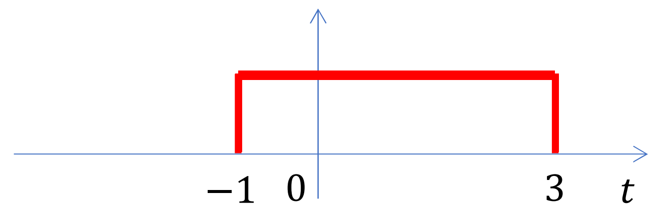


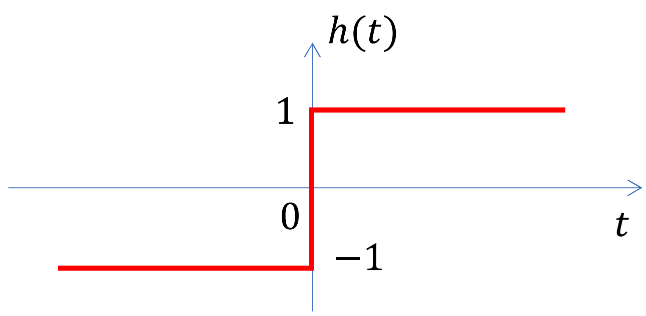
This relationship is casual & $y(t)$ depends only on values of $x(\tau)$ for $\tau \leq t$.

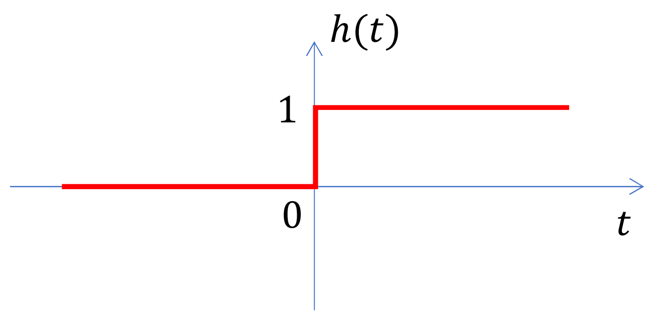
LTI Systems: Stability Test and Causality Test

Which of these four impulse responses are from LTI systems that are stable and/or causal?



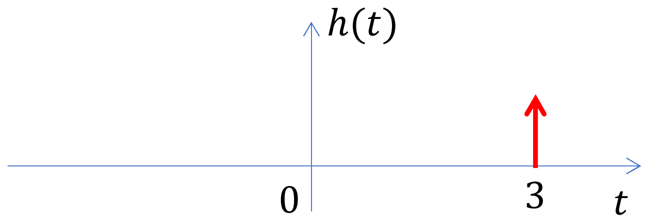




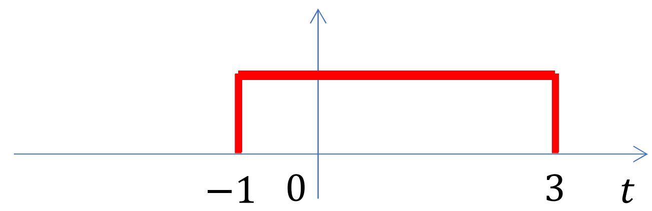


LTI Systems: Stability Test and Causality Test

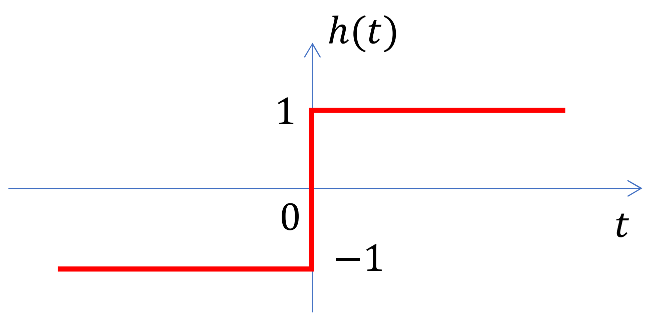
The following system is casual & BIBO stable. $\int_{-\infty}^{\infty}|h(\tau)|d\tau = 1$



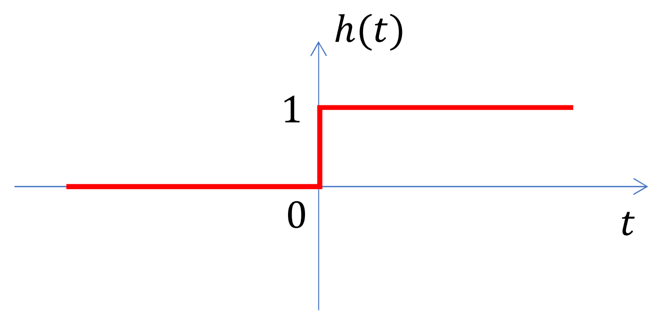
The following system is non-casual (starts at $t=-1$) & BIBO stable. $\int_{-\infty}^{\infty}|h(\tau)|d\tau = 4$



The following system is non-casual (starts at $t=-\infty$) & not BIBO stable. $\int_{-\infty}^{\infty}|h(\tau)|d\tau = \int_{-\infty}^{0}h(\tau)d\tau + \int_{0}^{\infty}h(\tau)d\tau = \infty$

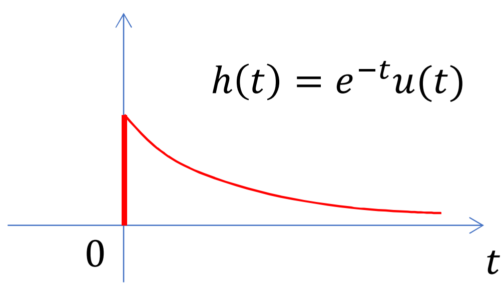


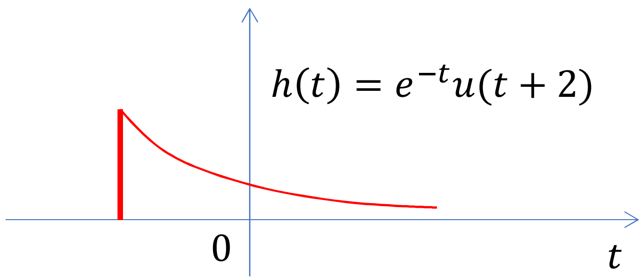
The following system is casual (starts at $t=0$) & not BIBO stable. $\int_{-\infty}^{\infty}|h(\tau)|d\tau = \int_{0}^{\infty}1d\tau = \infty$

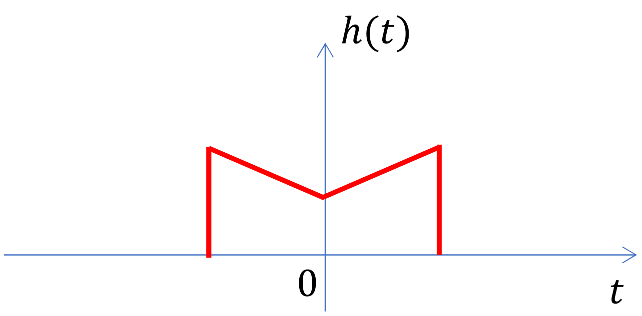


LTI Systems: Stability Test and Causality Test

The followings are impulse responses of LTI systems. Which ones are causal? which ones are BIBO stable?







Some Important Properties of Convolution

1- Commutative: $h(t) * x(t) = x(t) * h(t)$

2- Distribution: $h(t) * (x_1(t) + x_2(t)) = h(t) * x_1(t) + h(t) * x_2(t)$

3- Associative: $h(t) * (x_1(t) * x_2(t)) = (h(t) * x_1(t)) * x_2(t)$

4- Shift property: if $h(t) * x(t) = y(t)$ then

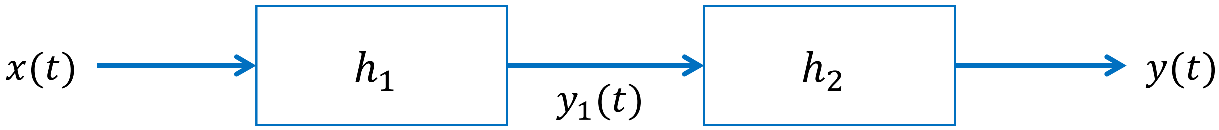
$$h(t - t_0) * x(t) = y(t - t_0)$$

$$h(t) * x(t - t_1) = y(t - t_1)$$

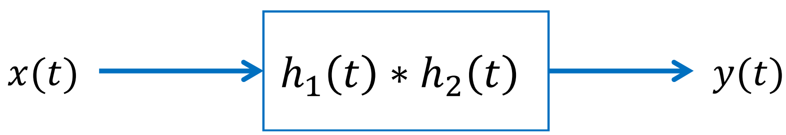
$$h(t - t_0) * x(t - t_1) = y(t - t_0 - t_1)$$

Interconnected Systems

Cascaded Systems:

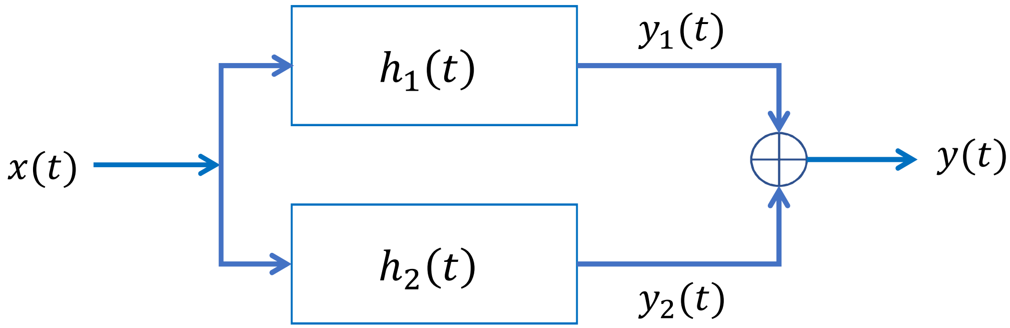


$$
\begin{aligned}
y(t) &= y_1(t) * h_{(}t) \\
&= (x(t) * h_1(t)) * h_2(t) \\
&= x(t) * \underbrace{(h_1(t) * h_2(t))}_{\text{using associativee property}}
\end{aligned}
$$

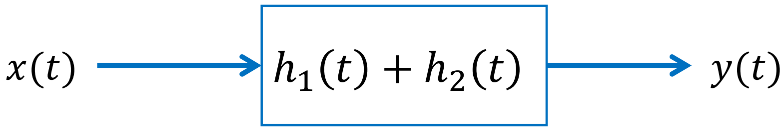


Interconnected Systems

Parallel Systems:

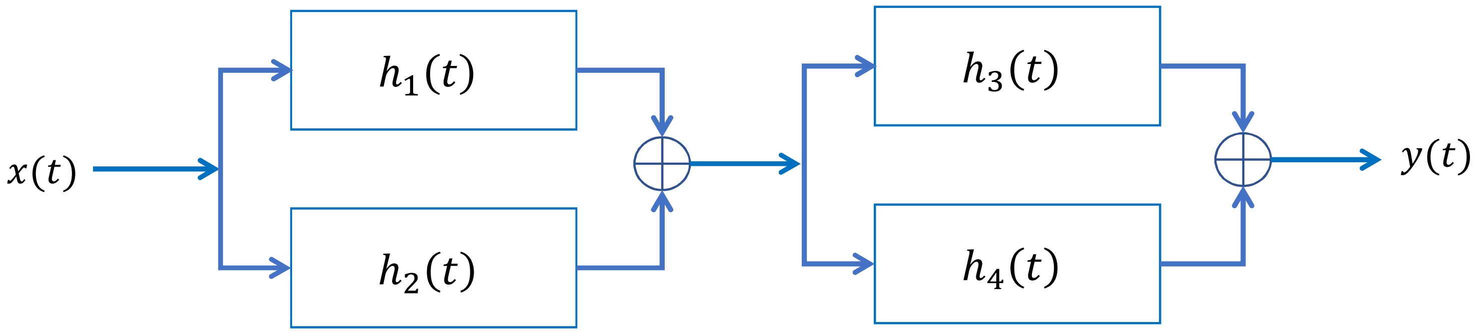


$$
\begin{aligned}
y(t) &= y_1(t) + y_2(t) \\
&= x(t) * h_1(t) + x(t) * h_2(t) \\
&= x(t) * \underbrace{(h_1(t) + h_2(t))}_{\text{using distributive property}}
\end{aligned}
$$



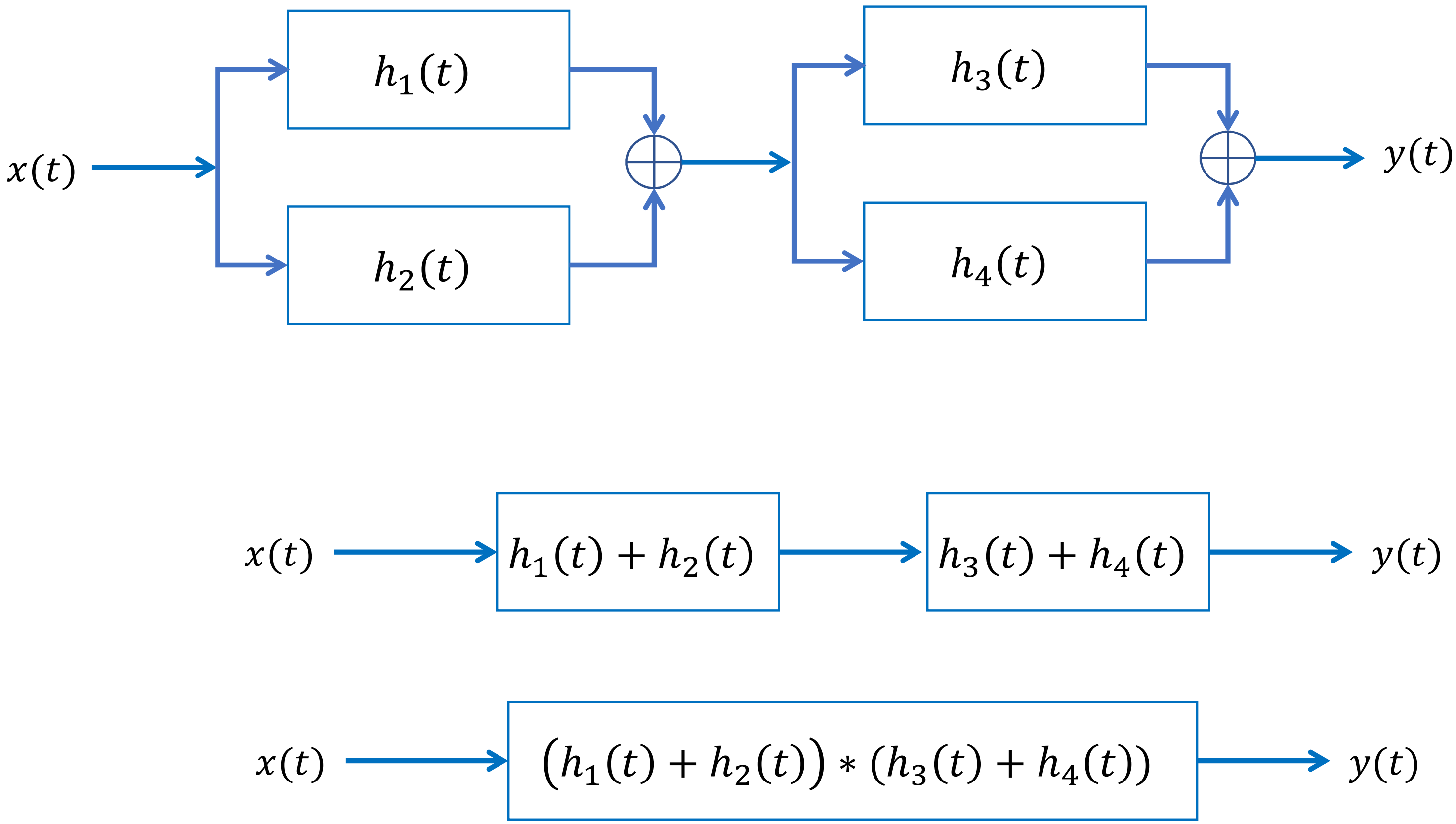
Interconnected Systems

Find the overall $h(t)$ of the following system:



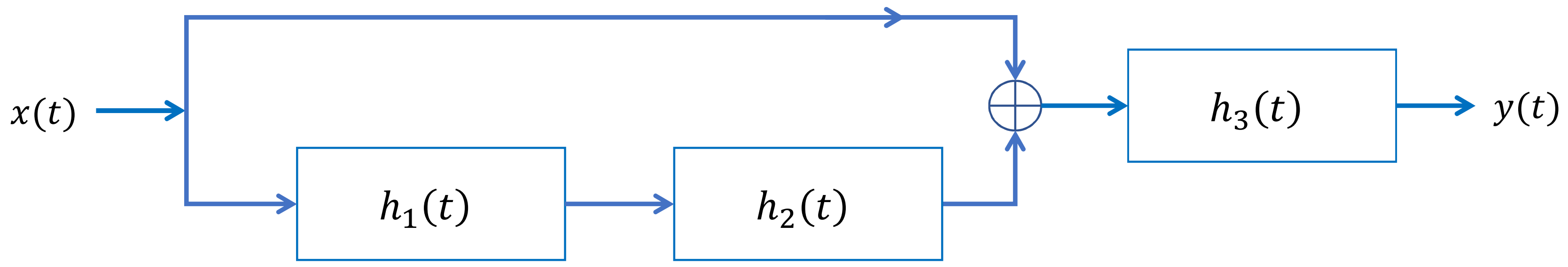
Interconnected Systems

Find the overall $h(t)$ of the following system:



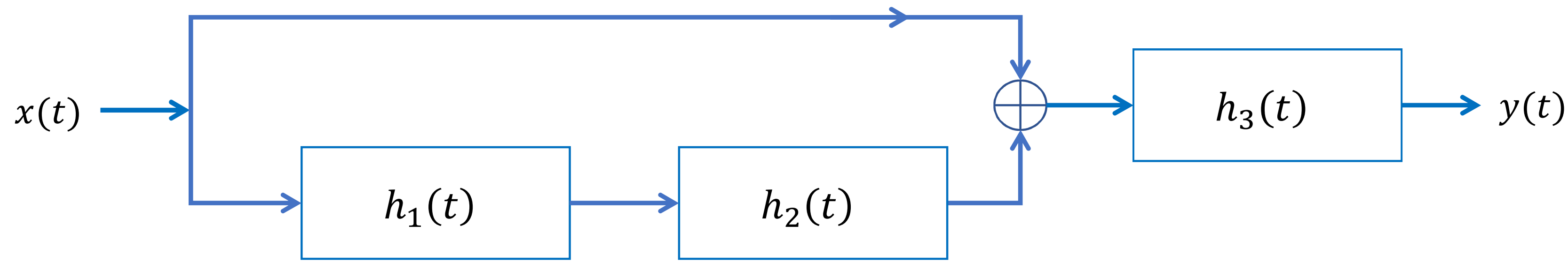
Interconnected Systems

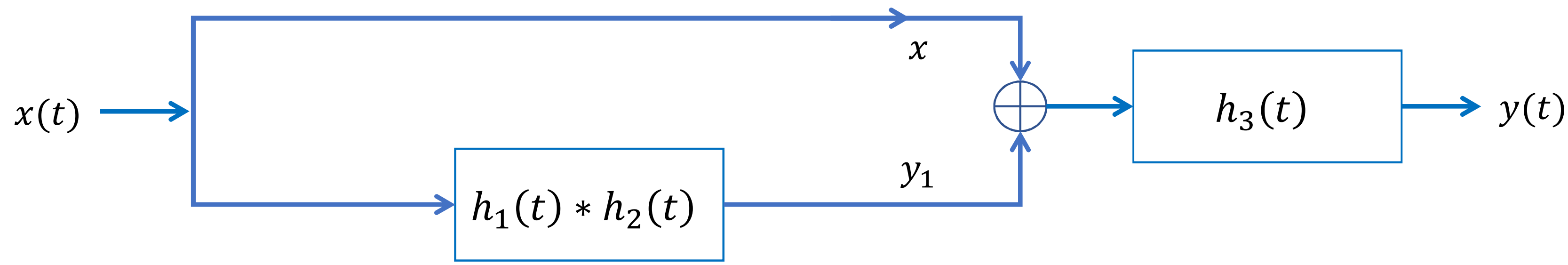
Find the overall $h(t)$ of the following system:



Interconnected Systems

Find the overall $h(t)$ of the following system:

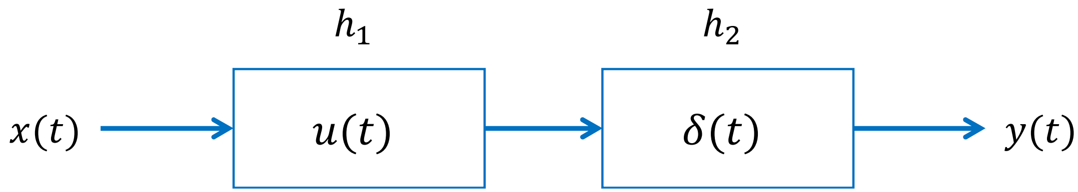




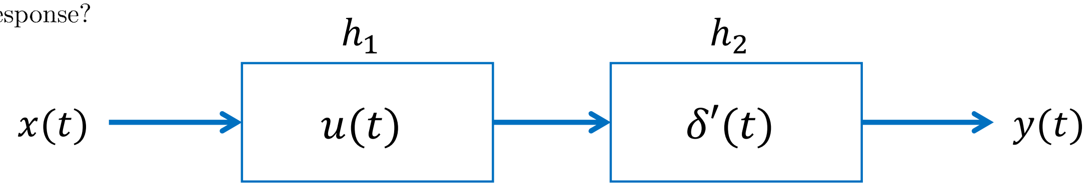
$$
\begin{aligned}
y(t) &= (x(t) + x(t) * h_1(t) * h_2(t)) * h_3(t) \\
&= x(t) * h_3(t) + x(t) * h_1(t) * h_2(t) * h_3(t) \\
&= x(t) * \underbrace{(h_3(t) + h_1(t) * h_2(t) * h_3(t))}_{\text{Overall system}}
\end{aligned}
$$

Interconnected Systems

What is the overall imupulse response?

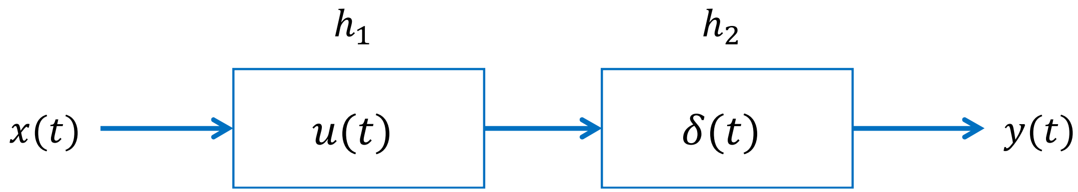


What is the overall imupulse response?



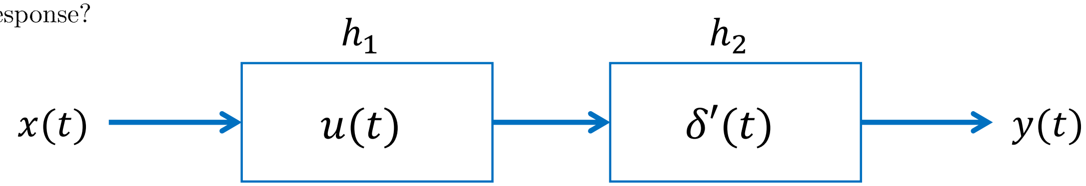
Interconnected Systems

What is the overall imupulse response?



$$h_1 * h_2 = u(t) * \delta(t) = u(t)$$

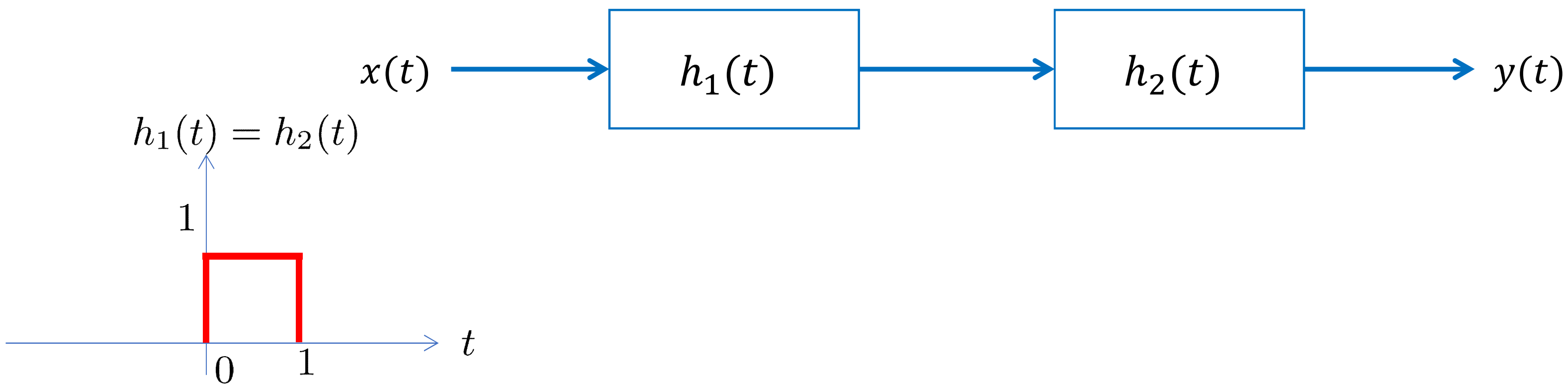
What is the overall imupulse response?



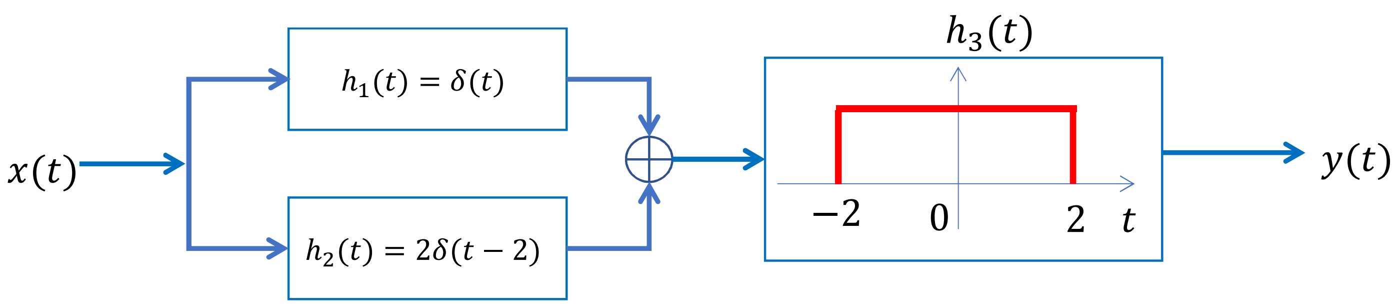
$$h_1 * h_2 = u(t) * \delta'(t) = \delta(t)$$

Interconnected Systems

Find the overall $h(t)$ of the following cascade system

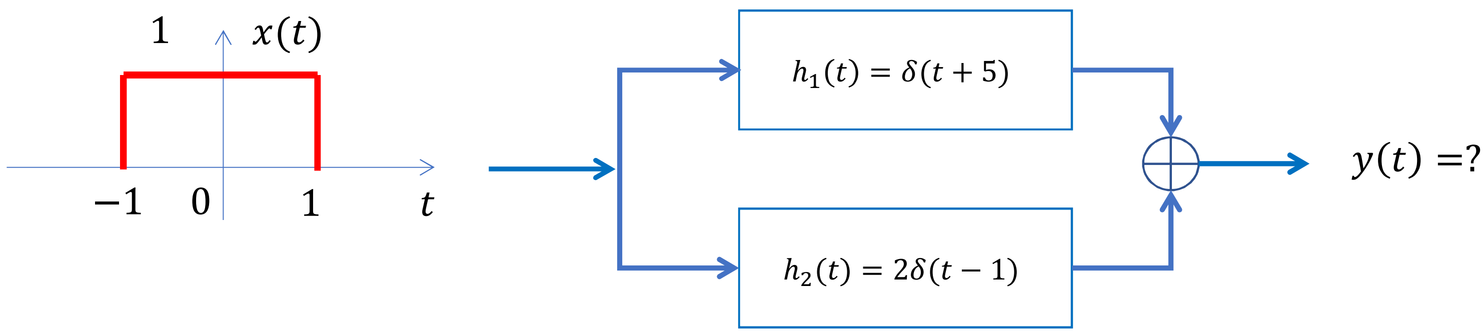


Find the overall $h(t)$ of the following cascade system

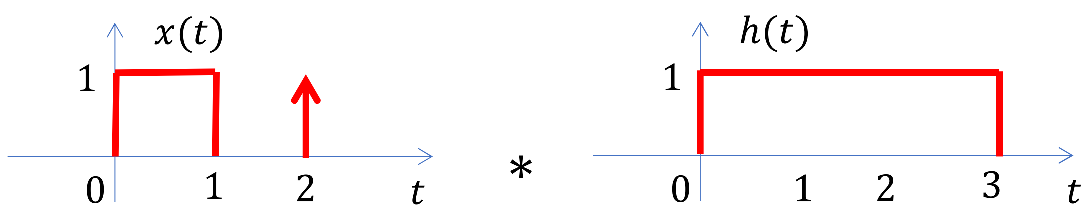


Interconnected Systems

Find the output:

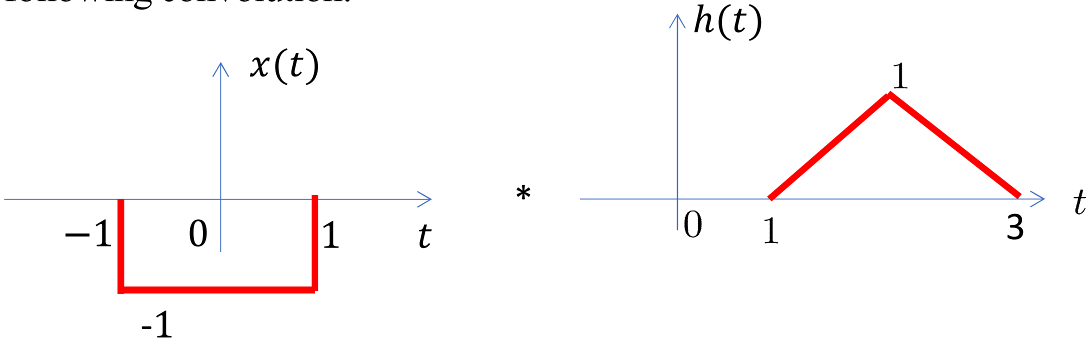


Find the convolution:

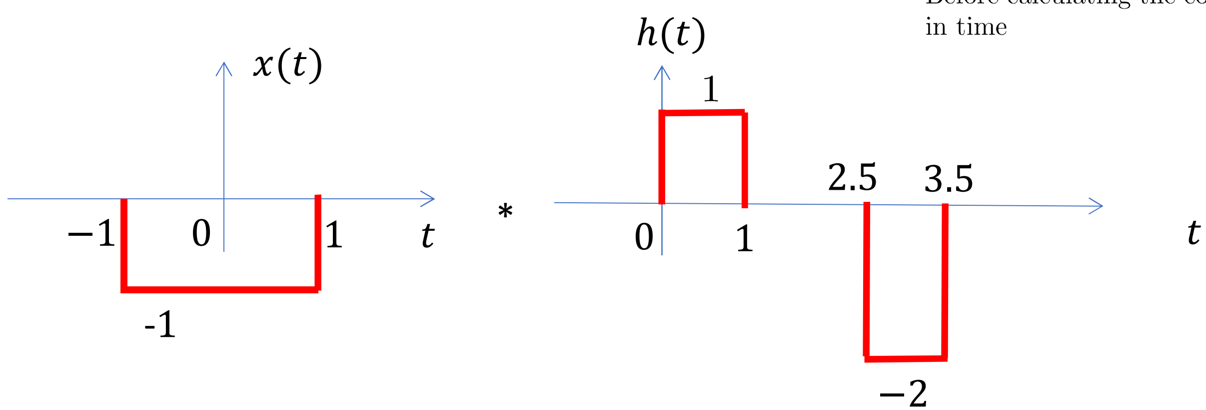


Convolution

Find the following convolution:



Before calculating the convolution find the boundaries of the convolution result in time



Use the GUI to evaluate your results

Impulse Response of LTIDE systems (Alternative Approach)

$$\left(D^N + a_1 D^{N-1} + ... + a_N\right) y(t) = \underbrace{\left(b_{N-M} D^M + b_{N-M-1} D^{M-1} + ... + b_N\right)}_{\text{Moving Average(MA) system}} x(t)$$

Step 1: Consider $h_1(t) = C(t)u(t)$ where $C(t)$ is the characteristic function of the above system.

Solve for coefficients of $h_1(t)$ such that

$$C(0) = 0,\ C'(0) = 0, \quad \cdots \quad , C^{(N-1)}(0) = 1$$

For example for the case of non repeated roots Characteristic function is $C(t) = (c_1 e^{\lambda_1 t} + ... + c_N e^{\lambda_N t})$

Step 2: To find $h(t)$ a linear combination of derivatives of $h_1(t)$ is calculated with b_i coefficients using the Moving Average (MA) structure:

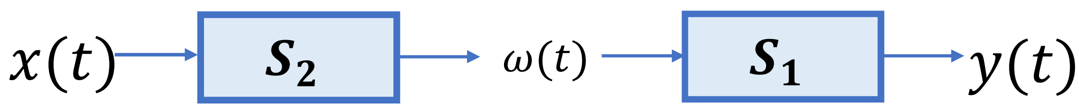
$$h_2(t) = (b_{N-M} D^M + ... + b_N) h_1(t)$$

In next slides we show why these two steps provide the system's impulse response.

Impulse Response of LTIDE systems (Alternative Approach)

$$\left(D^N + a_1 D^{N-1} + \dots + a_N\right) y(t) = \underbrace{\left(b_{N-M} D^M + b_{N-M-1} D^{M-1} + \dots + b_N\right)}_{\text{Moving Average(MA) system}} x(t)$$

$$= w(t)$$

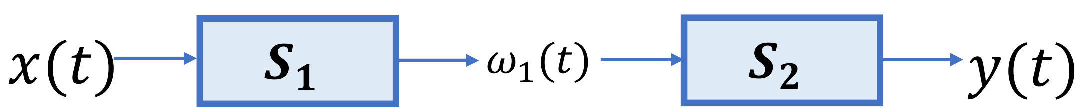


For $h_2(t)$: $w(t) = (b_{N-M} D^M + \dots + b_N) x(t)$

For $h_1(t)$: $\left(D^N + a_1 D^{N-1} + \dots + a_N\right) y(t) = w(t)$

Due to the associative property of convolution

$y(t) = x(t) * h_2(t) * h_1(2) = x(t) * h_1(t) * x_2(t)$



For $h_1(t)$: $\left(D^N + a_1 D^{N-1} + \dots + a_N\right) w_1(t) = x(t)$

For $h_2(t)$: $y(t) = (b_{N-M} D^M + \dots + b_N) w_1(t)$

First we find impulse response $h_1(t)$

Impulse Response of LTIDE systems (Alternative Approach)

For system S_1 we have

$$(D^N + a_1 D^{N-1} + ... + a_N)w_1(t) = x(t)$$

where $w_1(t)$ is the output. For impulse response $h_1(t)$ we have :

$$(D^N + a_1 D^{N-1} + ... + a_N)h_1(t) = \delta(t)$$

Since $b_0 = 0$ the impulse response has the following stucture:

$$h_1(t) = C(t)u(t), \qquad \text{where } \; C(t) \text{ is the Characteristic function}$$
$$h_1'(t) = C'(t)u(t) + C(0)\delta(t)$$
$$h_1''(t) = C''(t)u(t) + C'(0)\delta(t) + C(0)\delta'(t)$$
$$h_1'''(t) = C'''(t)u(t) + C''(0)\delta(t) + C'(0)\delta'(t) + C(0)\delta''(t)$$

and finally

$$h_1^{(N)}(t) = C^{(N)}(t)u(t) + C^{(N-1)}(0)\delta(t) + C^{(N-2)}\delta'(t) + ... + C(0)\delta^{(N-1)}(t)$$

Impulse Response of LTIDE systems (Alternative Approach)

$$(D^N + a_1 D^{N-1} + ... + a_N)h_1(t) = \delta(t)$$

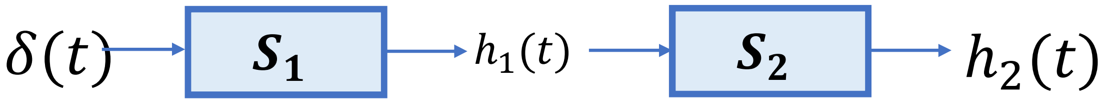
$$
\begin{aligned}
a_N h_1(t) + a_{N-1} h_1'(t) + ... = & \overbrace{\left[a_N C(t) + a_{N-1} C'(t) + ... + C^{(N)}(t)\right]}^{\text{already equal to zero!}} u(t) \\
& + \overbrace{\left[a_{N-1} C(0) + a_{N-2} C'(0) + ... + C^{(N-1)}(0)\right]}^{\text{This term should be equal to 1}} \delta(t) \\
& + \overbrace{\left[a_{N-2} C(0) + a_{N-3} C'(0) + ... + C^{(N-2)}(0)\right]}^{\text{This term should be equal to zero}} \delta'(t) \\
& + \vdots \\
& + \overbrace{[a_1 C(0) + C'(0)]}^{\text{This term should be equal to zero}} \quad \delta^{N-2}(t) \\
& + \overbrace{C(0)}^{\text{This term should be equal to zero}} \quad \delta^{N-1}(t) \\
= & \ \delta(t)
\end{aligned}
$$

From the above equation we should have:

$$C(0) = 0,\ C'(0) = 0,\quad \cdots \quad , C^{(N-1)}(0) = 1$$

Impulse Response of LTIDE systems (Alternative Approach)

For the overall $h(t)$ we have



For $h_2(t)$: $h_2(t) = (b_{N-M}D^M + ... + b_N)h_1(t)$

So to find $h(t)$ a linear combination of derivatives of $h_1(t)$ is calculated with b_i coefficients.