Signals and Systems I

Topic 8

Last Lecture: Fourier Series

Fourier Series

- $e^{j\omega t}$

- $\sin(\omega t + \theta)$
- $\cos(\omega t + \theta)$
- Periodic pulse in time → Fourier Series in the form of Sinc

Today
- Trigonometric Fourier Series & Exponential Fourier Series
- LTI Systems, Eigen Function of LTI System (e^{st})
- LTI Systems & Fourier Series
- More Examples
- Some properties of FS

Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 n t} \qquad \text{(Fourier series)} \Longleftarrow Synthesis$$

$$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\omega_0 n t} dt \qquad \text{(Finds } D_n \text{ from } x(t)) \Longleftarrow Analysis$$

$$D_n e^{jn\omega_0 t}$$

Componencts of Fourier Series are periodic spirals in form of $e^{jn\omega_0 t}$ which is a periodic spiral with frequency $n\omega_0$.

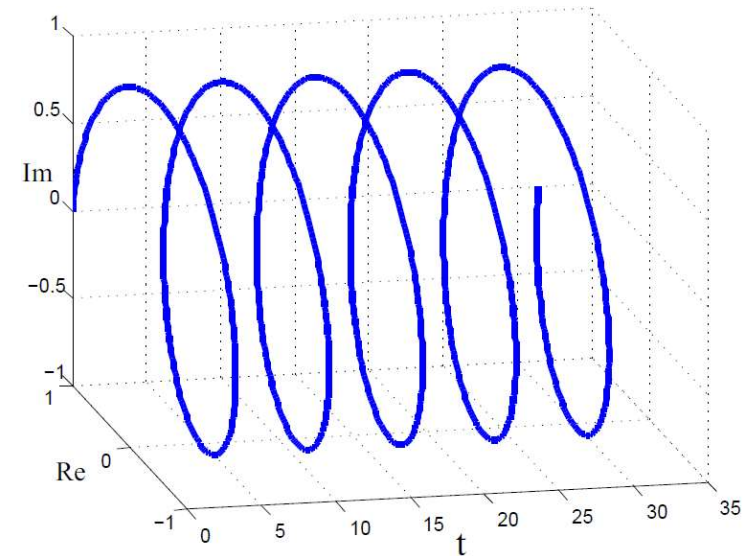Each spiral is then rotated by angle of D_n and amplified by $|D_n|$

$$x_1(t) = j e^{-j5t}$$

$$x_2(t) = 2 e^{j\frac{\pi}{3}} e^{j2\pi t}$$

$$x_3(t) = -2 e^{j\frac{5}{6}\pi t}$$

$$\omega_0 = -5, \ |D_{-1}| = 1, \ \angle(D_{-1}) = \frac{\pi}{2}$$

$$\omega_0 = 2\pi, \ |D_1| = 2, \ \angle(D_1) = \frac{\pi}{3}$$

$$\omega_0 = \frac{5}{6}\pi, \ |D_1| = 2, \ \angle(D_1) = \pi$$

Fourier Series

Euler Formulas:

$$\cos(\alpha t + \beta) = \frac{e^{j(\alpha t + \beta)} + e^{-j(\alpha t + \beta)}}{2}$$

$$\sin(\alpha t + \beta) = \frac{e^{j(\alpha t + \beta)} - e^{-j(\alpha t + \beta)}}{2j}$$

Associated FS coefficients: $\omega_0 = \alpha$

$$\cos(\alpha t + \beta) = \underbrace{\frac{1}{2}e^{\beta}}_{D_1} e^{j(\alpha t)} + \underbrace{\frac{1}{2}e^{j\beta}}_{D_{-1}} e^{j(-\alpha t)}$$

$$\sin(\alpha t + \beta) = \underbrace{\frac{1}{j2}e^{\beta}}_{D_1} e^{j(\alpha t)} + \underbrace{\frac{-1}{2j}e^{j\beta}}_{D_{-1}} e^{j(-\alpha t)}$$

$\rightarrow$ One Spiral = One Coefficient

$\rightarrow$ One Sine/Cosine = Two Coefficients (one positive and one negative: $D_1 \& D_{-1}$

Trigonometric Fourier Series and Exponential Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 n t} \qquad \text{(Exponential Fourier series)}$$

$$x(t) = D_0 + \underbrace{D_1 e^{j\omega_0 t} + D_{-1} e^{-j\omega_0 t}} + \underbrace{D_2 e^{j2\omega_0 t} + D_{-2} e^{-j2\omega_0 t}} + \underbrace{D_3 e^{j3\omega_0 t} + D_{-3} e^{-j3\omega_0 t}} + \cdots$$

$$x(t) = a_0 + \underbrace{a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t)} + \underbrace{a_2 \cos(2\omega_0 t) + b_2 \sin(2\omega_0 t)} + \underbrace{a_3 \cos(3\omega_0 t) + b_3 \sin(3\omega_0 t)}$$

Euler
formula

$$a_1\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) + b_1\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)$$

$$\left(\frac{a_1}{2} + \frac{b_1}{2j}\right)e^{j\omega_0 t} + \left(\frac{a_1}{2} - \frac{b_1}{2j}\right)e^{-j\omega_0 t})$$

$$\text{for} \quad n > 0 \quad \begin{cases} D_0 = a_0 \\ D_n = \frac{a_n}{2} + \frac{b_n}{2j} \\ D_{-n} = \frac{a_n}{2} - \frac{b_n}{2j} \end{cases} \overset{\text{or}}{\Longrightarrow} \begin{cases} D_n + D_{-n} = a_n \\ D_n - D_{-n} = \frac{b_n}{j} \end{cases}$$
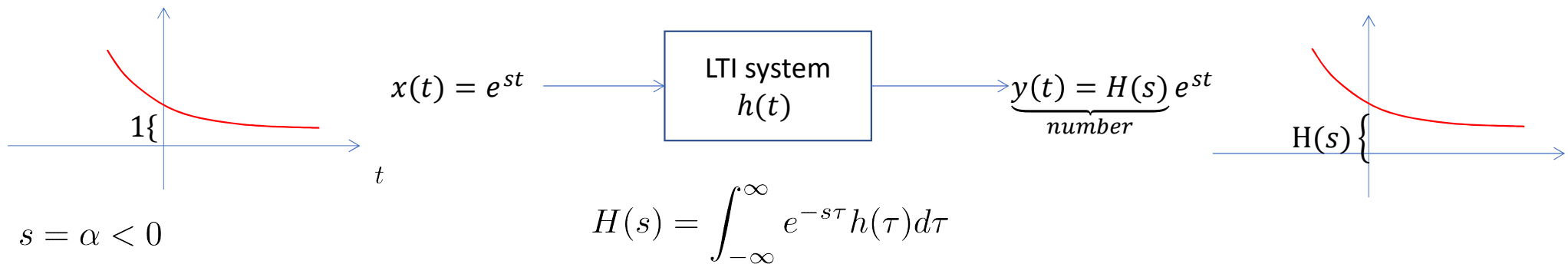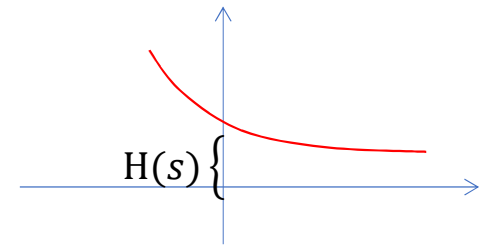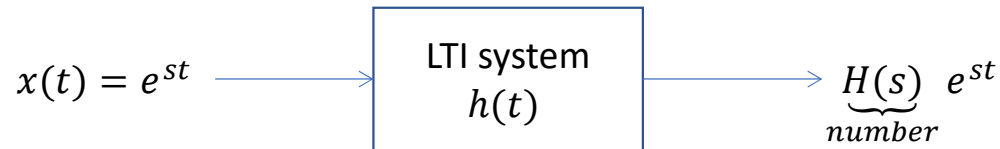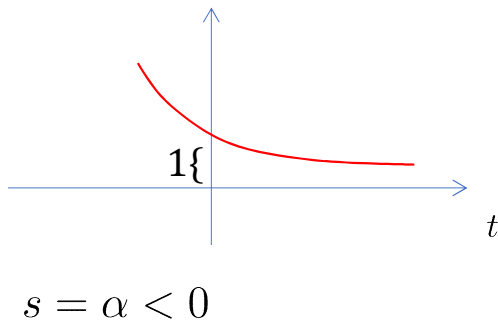
Simple Example:

$$x(t) = cos(\omega_0 t)$$

Trigonometric FS: $\qquad a_0 = 0, a_1 = 1, a_2 = 0, ..., b_n = 0, \quad \forall n$

Exponential FS: $\qquad D_0 = 0, D_1 = \frac{1}{2}, D_{-1} = \frac{1}{2}$

Exponentials e^{st} and LTI Systems

$$x(t) = e^{st}$$

$$s = \alpha < 0$$

LTI system
$h(t)$

$$\underbrace{y(t) = H(s)}_{number} e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau)\,d\tau$$

H(s)

Exponentials e^{st} and LTI Systems

$x(t) = e^{st}$ → LTI system $h(t)$ → $\underbrace{H(s)}_{number} \; e^{st}$

$1\{$

t

$s = \alpha < 0$

$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$

$H(s)\{$

Why?

$$y(t) = x(t) * h(t)$$
$$= e^{st} * h(t)$$
$$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

e^{st} is <u>eigenfunction</u> of LTI systems.

$$= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau$$
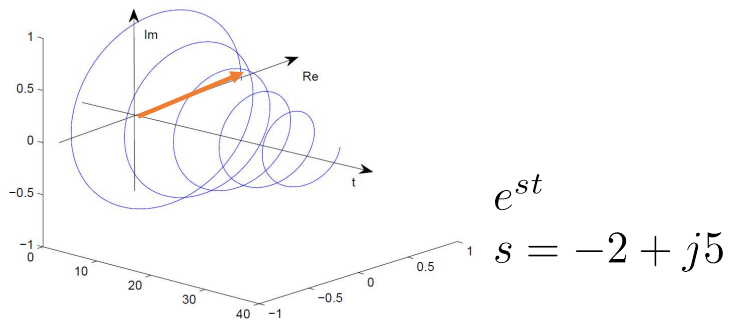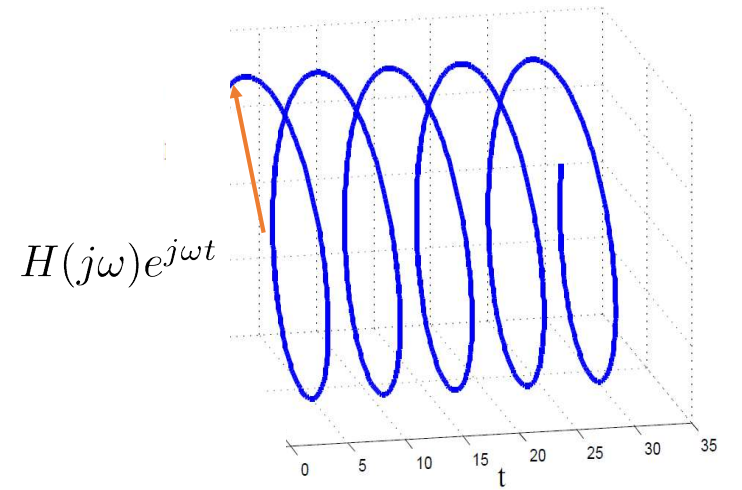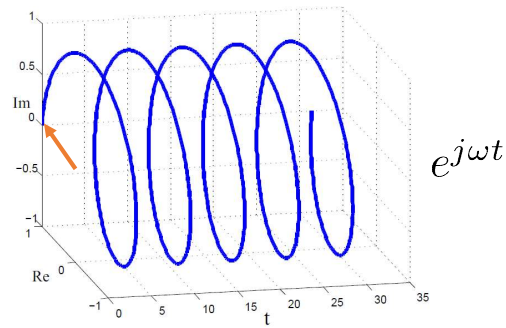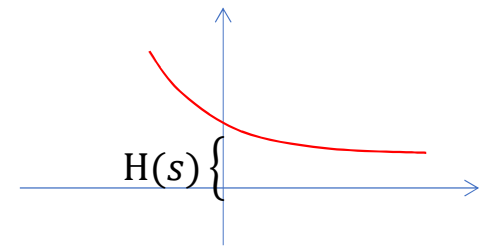
And $H(s)$ is its <u>eigenvalue</u>.

Reminder of eigenvalue and eigenvector for Matrix A transformation: $y = Ax$ where input is vector x and output is vector y. Eigenvalue is the a v with eigenvalue λ such that $\lambda v = Av$

$$= e^{st} \int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau$$

$$= e^{st} H(s)$$

Exponentials e^{st} and LTI Systems

$x(t) = e^{st}$

$\boxed{\begin{array}{c} \text{LTI system} \\ h(t) \end{array}}$

$\underbrace{H(s)}_{number}\ e^{st}$

$1\{$ $s = \alpha < 0$ t

$H(s)\{$

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

$e^{j\omega t}$

$H(j\omega)e^{j\omega t}$

e^{st}
$s = -2 + j5$

$H(s)e^{st}$
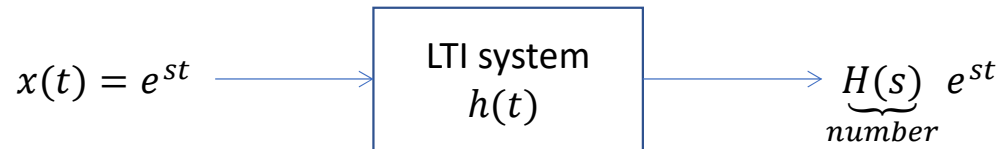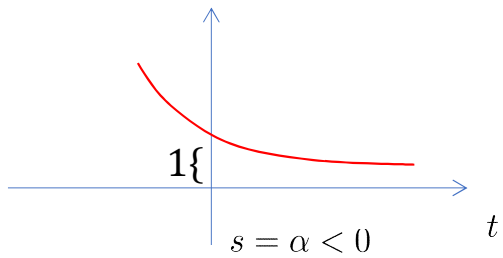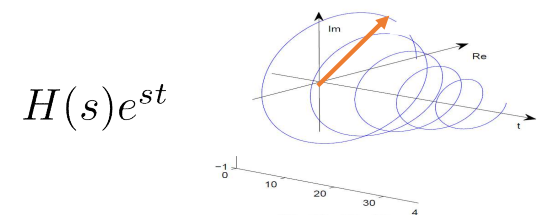
Periodic Signals and LTI Systems

Consider the case for $s = j\omega$ (Pure Imaginary)

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\begin{array}{c} \text{LTI system} \\ h(t) \end{array}} \longrightarrow \underbrace{H(j\omega)}_{number} e^{j\omega t}$$
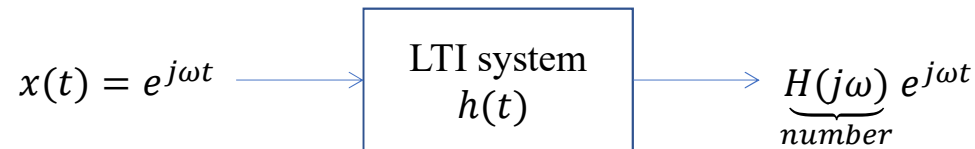
$$H(j\omega) = \int e^{-j\omega\tau} h(\tau) d\tau$$

This property can be used to find the output of LTI systems to periodic inputs.
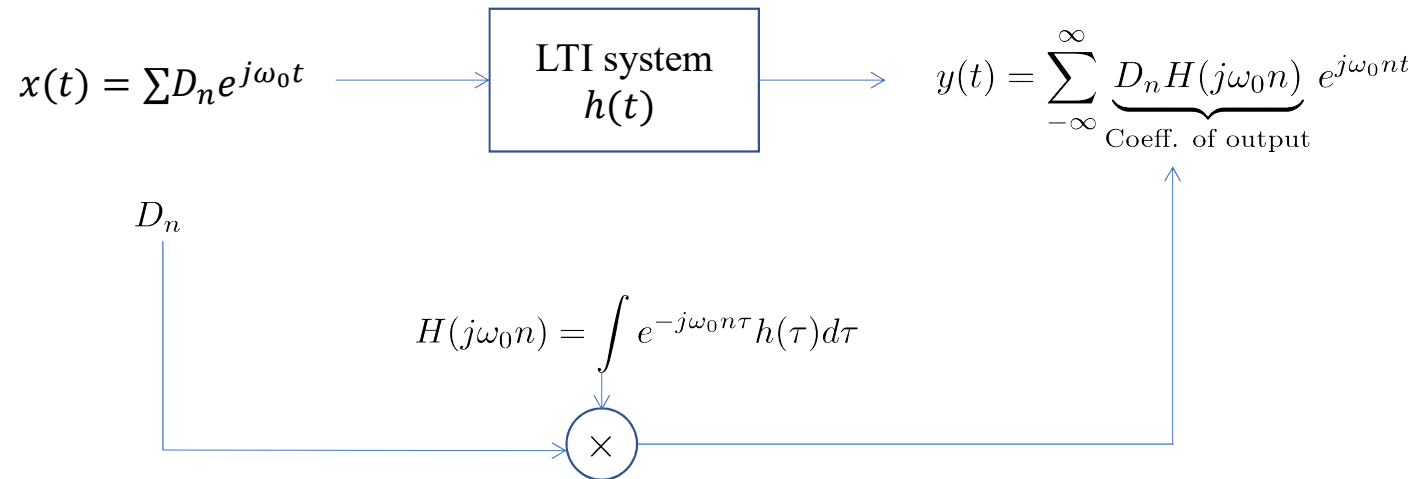If $x(t)$ is periodic: $x(t) = \sum_{-\infty}^{\infty} D_n e^{j\omega_0 n t}$

$$x(t) = \sum D_n e^{j\omega_0 t} \longrightarrow \boxed{\begin{array}{c} \text{LTI system} \\ h(t) \end{array}} \longrightarrow y(t) =?$$
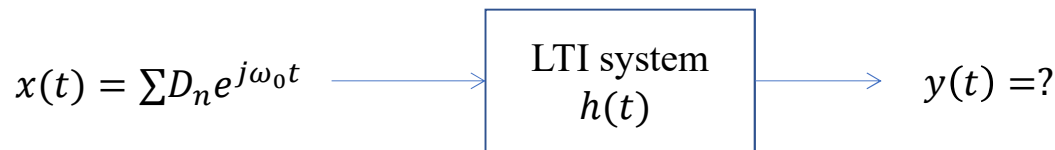
$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \left(\sum_{-\infty}^{\infty} D_n e^{j\omega_0 n t} \right) * h(t) \\ &= \sum_{-\infty}^{\infty} D_n \left(e^{j\omega_0 n t} * h(t) \right) \\ &= \sum_{-\infty}^{\infty} D_n \underbrace{H(j\omega_0 n)}_{\text{Coeff. of output}} e^{j\omega_0 n t} \end{aligned}$$

Convolution is transformed to
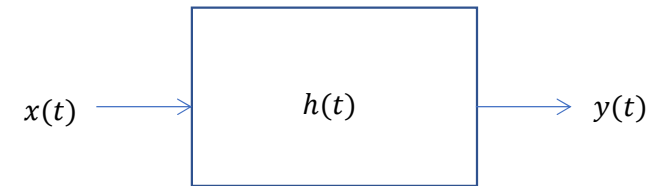"product of eigenvalues and Fourier Series Coefficients!"

Periodic Signals and LTI Systems

$$x(t) = \sum D_n e^{j\omega_0 t}$$

$$\boxed{\begin{array}{c} \text{LTI system} \\ h(t) \end{array}}$$

$$y(t) = \sum_{-\infty}^{\infty} \underbrace{D_n H(j\omega_0 n)}_{\text{Coeff. of output}} e^{j\omega_0 nt}$$

$$D_n$$

$$H(j\omega_0 n) = \int e^{-j\omega_0 n\tau} h(\tau) d\tau$$

$$\times$$

It is faster to find the Convolution results using $H(j\omega_0 n)$s and Fourier Series Coefficients

Periodic Signals and LTI Systems

Output is also periodic with coefficients $D_n H(j\omega_0 n)$.

$$x(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t)$$

To Find output of an LTI system to a periodic input $x(t)$:

1- Find D_n coefficients of Fourier series of $x(t)$.

$$x(t) = \sum D_n e^{j\omega_0 n t}$$

2- For ω_0, $2\omega_0$, $3\omega_0, ...,$ $n\omega_0$ (where ω_0 is the fundamental freq. of $x(t)$) find $H(j\omega_0 n)$:

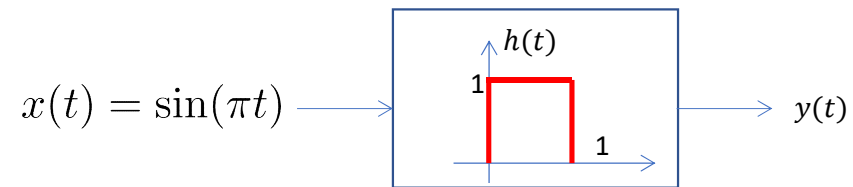$$H(j\omega_0 n) = \int_{-\infty}^{\infty} e^{-j\omega_0 n \tau} h(\tau) d\tau$$

3- Coefficients of $y(t)$ (output of the system to $x(t)$) are $D_n H(j\omega_0 n)$ and the output is as follows:

$$y(t) = \sum D_n H(j\omega_0 n) e^{j\omega_0 n t}$$
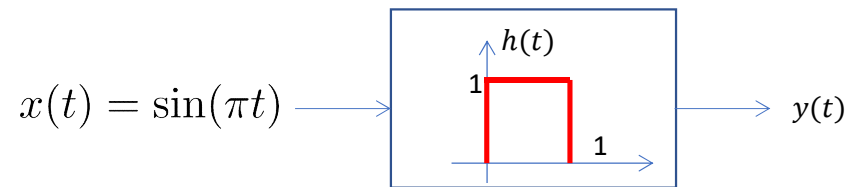
Output of LTI system to periodic signal has the $\underline{same}$ period

Periodic Signals and LTI Systems

Example: Find the output of the following LTI system to $x(t) = \sin(\pi t)$?

Periodic Signals and LTI Systems

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$$x(t) = \sin(\pi t) \longrightarrow \boxed{\begin{array}{c} h(t) \\ \end{array}} \longrightarrow y(t)$$

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Periodic Signals and LTI Systems

$$x(t) = \sin(\pi t) \longrightarrow \boxed{\begin{array}{c} h(t) \\ 1 \\ \hline \quad \\ 1 \end{array}} \longrightarrow y(t)$$
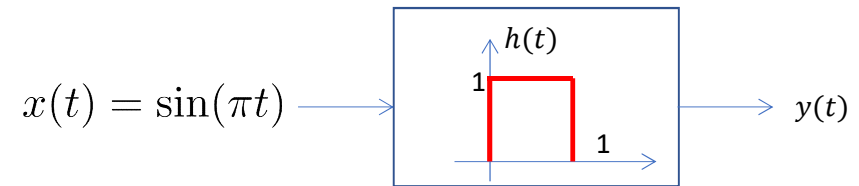
1- $x(t)$ is periodic ($\omega_0 = \pi$)

$$x(t) = sin(\pi t) = \underbrace{\frac{1}{2j}}_{D_1^x} e^{j\pi t} - \underbrace{\frac{1}{2j}}_{D_{-1}^x} e^{-j\pi t}$$
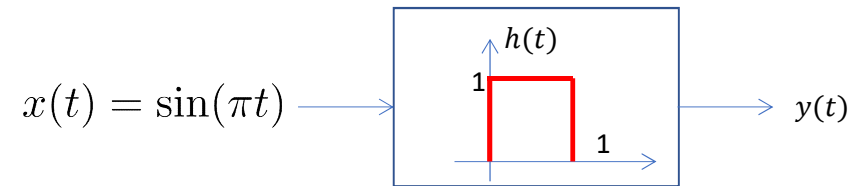
2- Since only D_1^x & D_{-1}^x are non-zero, we have to find $H(j\omega_0)$ & $H(-j\omega_0)$:

$$H(j\omega_0) = H(j\pi) = \int_{-\infty}^{\infty} e^{-j\pi\tau} h(\tau) d\tau$$

$$= \int_0^1 e^{-j\pi\tau} 1 d\tau$$

$$= \frac{e^{-j\pi\tau}}{-j\pi}\Big|_0^1 = \frac{1 - e^{-j\pi}}{j\pi} = \frac{2}{j\pi}$$

$$H(-j\pi) = \int_0^1 e^{j\pi\tau} 1 d\tau$$

$$= \frac{e^{j\pi}-1}{j\pi} = \frac{-2}{j\pi}$$

Periodic Signals and LTI Systems

$$x(t) = \sin(\pi t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t)$$

1- $x(t)$ is periodic ($\omega_0 = \pi$)

$$x(t) = \sin(\pi t) = \underbrace{\frac{1}{2j}}_{D_1^x} e^{j\pi t} - \underbrace{\frac{1}{2j}}_{D_{-1}^x} e^{-j\pi t}$$

3- Coefficients of $y(t)$ are:

2- Since only D_1^x & D_{-1}^x are non-zero, we have to find $H(j\omega_0)$ & $H(-j\omega_0)$:

$$D_1^y = D_1^x . H(j\pi) = \frac{1}{2j} \cdot \frac{2}{j\pi} = \frac{-1}{\pi}$$

$$H(j\omega_0) = H(j\pi) = \int_{-\infty}^{\infty} e^{-j\pi\tau} h(\tau) d\tau$$

$$D_{-1}^y = D_{-1}^x . H(-j\pi) = \frac{-1}{2j} \cdot \frac{-2}{j\pi} = \frac{-1}{\pi}$$

$$= \int_0^1 e^{-j\pi\tau} 1 d\tau$$

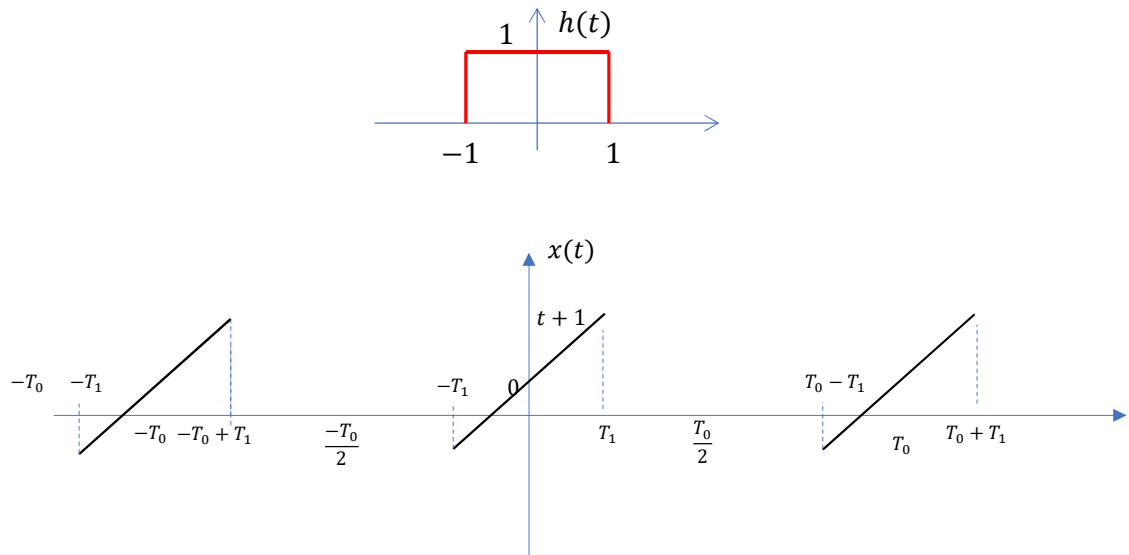$$= \frac{e^{-j\pi\tau}}{-j\pi} \bigg|_0^1 = \frac{1 - e^{-j\pi}}{j\pi} = \frac{2}{j\pi}$$

$$y(t) = \sum D_n^y e^{j\omega_0 n t}, \quad (\omega_0 = \pi)$$

$$= D_1^y e^{j\pi t} + D_{-1}^y e^{-j\pi t}$$

$$H(-j\pi) = \int_0^1 e^{j\pi\tau} 1 d\tau$$

$$= \frac{-1}{\pi} \left(e^{j\pi t} + e^{-j\pi t} \right)$$

$$= \frac{e^{j\pi} - 1}{j\pi} = \frac{-2}{j\pi}$$

$$= \frac{-2}{\pi} \cos(\pi t)$$

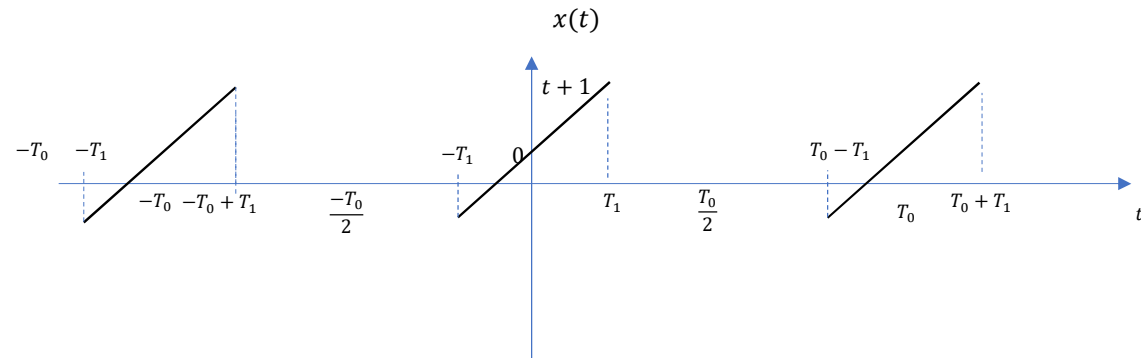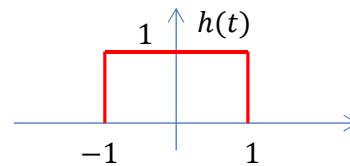Output of LTI system to periodic signal has the same period

Periodic Signals and LTI Systems

Example: Find Fourier series of $x(t)$ and use it to find $y(t) = x(t) * h(t)$ where $h(t) = u(t+1) - u(t-1)$. Solve for when $T_1 = 2$ and $T_0 = 4$.

Periodic Signals and LTI Systems

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To Find output of an LTI system to a periodic input $x(t)$:

1- Find D_n coefficients of Fourier series of $x(t)$.

$$x(t) = \sum D_n e^{j\omega_0 n t}$$

2- For $\omega_0,\ 2\omega_0,\ 3\omega_0, ...,\ n\omega_0$ (where ω_0 is the fundamental freq. of $x(t)$) find $H(j\omega_0 n)$:
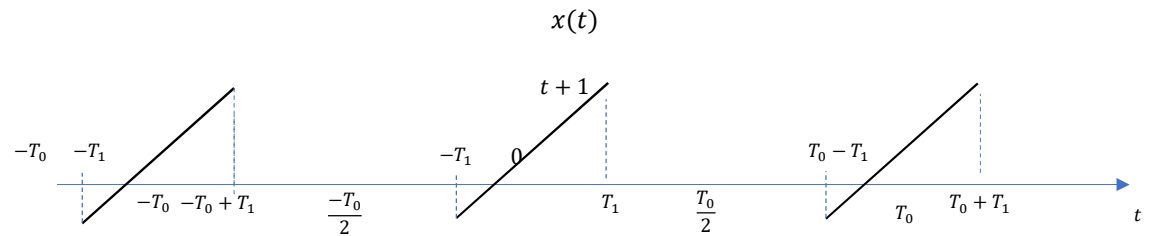
$$H(j\omega_0 n) = \int_{-\infty}^{\infty} e^{-j\omega_0 n \tau} h(\tau) d\tau$$

3- Coefficients of $y(t)$ (output of the system to $x(t)$) are $D_n H(j\omega_0 n)$ and the output is as follows:

$$y(t) = \sum D_n H(j\omega_0 n) e^{j\omega_0 n t}$$

Periodic Signals and LTI Systems

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt$$

$x(t)$

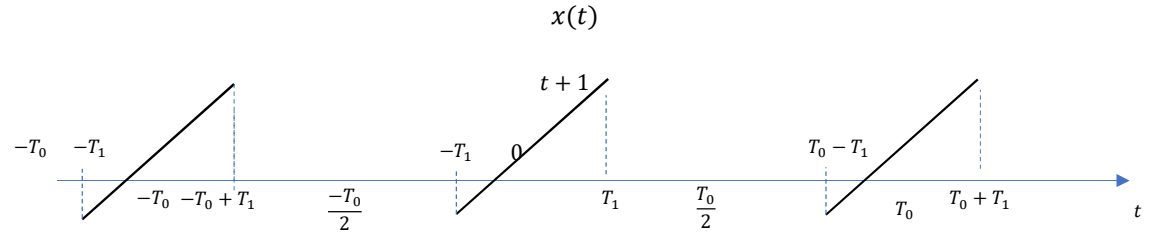# Periodic Signals and LTI Systems

$$x(t)$$

First find D_n s for $x(t)$:

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t)e^{-j\omega_0 nt}dt, \ \omega_0 = \frac{2\pi}{T_0}$$

$$D_n = \frac{1}{T_0} \int_{-T_1}^{T_1} (t+1)e^{-j\omega_0 nt}dt = \underbrace{\frac{1}{T_0} \int_{-T_1}^{T_1} te^{-j\omega_0 nt}dt}_{(1)} + \underbrace{\frac{1}{T_0} \int_{-T_1}^{T_1} e^{-j\omega_0 nt}dt}_{(2)}$$

(2): Calculation of integral in (2) for $n \neq 0$:

$$= \frac{1}{T_0} \frac{1}{-j\omega_0 n} e^{-j\omega_0 nt}\Big|_{-T_1}^{T_1} = \frac{1}{j\omega_0 nT_0}\left(e^{j\omega_0 nT_1} - e^{-j\omega_0 nT_1}\right) = \frac{2}{\omega_0 nT_0}sin(\omega_0 nT_1) = \alpha_n$$

Calculation of integral in (2) for $n = 0$

$$= \frac{1}{T_0} \int_{-T_1}^{T_1} 1dt = \frac{2T_1}{T_0} = \alpha_0$$

(1): In this step we use integration by part

$$\int_a^b \frac{d(uv)}{dt}dt = \int_a^b u\frac{dv}{dt}dt + \int_a^b v\frac{du}{dt}dt \Rightarrow UV\Big|_a^b = \int_a^b u\frac{dv}{dt}dt + \int_a^b v\frac{du}{dt}$$

Consider the following substitutions:

$$\begin{cases} u = t \rightarrow \frac{du}{dt} = 1 \\ \frac{dv}{dt} = e^{-j\omega_0 nt} \rightarrow v = \frac{e^{-j\omega_0 nt}}{-j\omega_0 n} \end{cases}$$

Now we can write:

$$\Rightarrow \frac{1}{T_0}\left(\int_{-T_1}^{T_1} te^{-j\omega_0 nt}dt = t.\frac{e^{-j\omega_0 nt}}{j\omega_0 n}\Big|_{-T_1}^{T_1} - \int_{-T_1}^{T_1} \frac{e^{-j\omega_0 nt}}{j\omega_0 n}.\underbrace{1}_{\frac{du}{dt}}dt\right)$$

$$= \frac{1}{T_0}\left(\frac{T_1 e^{-j\omega_0 nT_1} + T_1 e^{j\omega_0 nT_1}}{-j\omega_0 n} + \frac{e^{-j\omega_0 nt}}{(-j\omega_0 n)^2}\Big|_{-T_1}^{T_1}\right)$$

$$= \frac{1}{T_0}\left(\frac{j2T_1}{\omega_0 n}cos(\omega_0 nT_1) + j\frac{2sin(\omega_0 nT_1)}{\omega_0^2 n^2}\right)$$

$$= j\left(\frac{2T_1}{2\pi n}cos(\omega_0 nT_1) + \frac{2sin(\omega_0 nT_1)}{T_0\omega_0^2 n^2}\right), \qquad T_0\omega_0 = 2\pi$$

$$= j\beta_n$$

$$D_n = \alpha_n + j\beta_n$$

Periodic Signals and LTI Systems

For $T_1 = 2$ and $T_0 = 4$

$$x(t)$$

$$T_1 = 2$$
$$T_0 = 4, \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$$

(graph of $x(t)$, labeled $t+1$, with t axis marked at $-4, -2, 0, 2, 4$)

$$\alpha_n = \frac{2}{\omega_0 n T_0} sin(\omega_0 n T_1)$$

$$= \frac{2}{2\pi n} sin\left(\frac{\pi}{2} . 2 . n\right)$$

$$= \frac{1}{n\pi} sin(n\pi)$$

$$= \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$

For $n \neq 0$

$$\beta_n = \left(\frac{2T_1}{2\pi n} cos(\omega_0 n T_1) + \frac{2 sin(\omega_0 n T_1)}{T_0 \omega_0^2 n^2}\right), \qquad T_0 \omega_0 = 2\pi$$

$$= \frac{1}{4}\left(\frac{2.2}{\frac{\pi}{2} n} cos(\pi n) + \frac{2}{\omega_0^2 n^2} sin(\pi n)\right) = 2\frac{(-1)^n}{\pi n}$$

For $n = 0$

$$\beta_0 = \frac{1}{4}\int_{-2}^{2} t \, dt = 0$$

$$sin(\pi n) = 0$$
$$cos(\pi n) = (-1)^n$$

$$D_n = \begin{cases} 1 & n = 0 \\ \frac{j2(-1)^n}{\pi n} & n \neq 0 \end{cases}$$
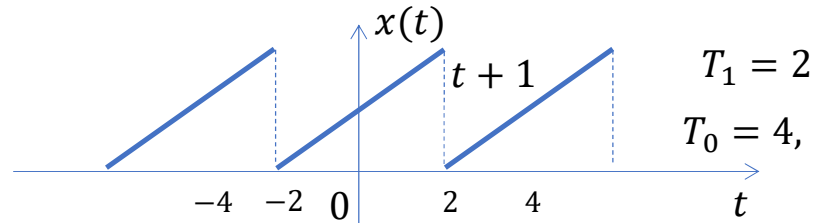
Periodic Signals and LTI Systems
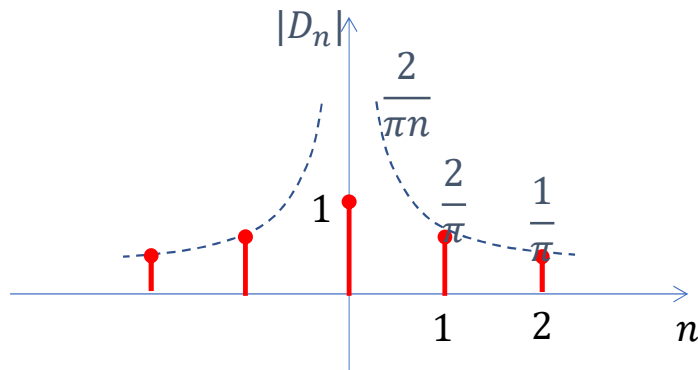
$$D_n = \begin{cases} 1 & n = 0 \\ \frac{j2(-1)^n}{\pi n} & n \neq 0 \end{cases}$$

$$|D_n| = \begin{cases} 1 & n = 0 \\ \frac{2}{\pi|n|} & n \neq 0 \end{cases} \Rightarrow \angle D_n = \begin{cases} 0 & n = 0 \\ \frac{\pi}{2} & (n > 0, \; n = \text{even}) \text{ and } (n < 0, \; n = \text{odd}) \\ \frac{-\pi}{2} & (n > 0, \; n = \text{odd}) \text{ and } (n < 0, \; n = \text{even}) \end{cases}$$
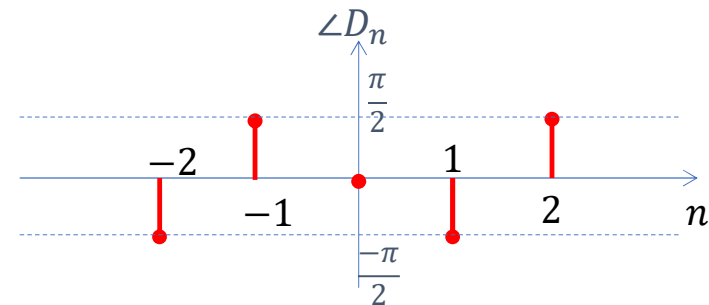
Periodic Signals and LTI Systems

Passing $x(t)$ through LTI system with given $h(t)$.

$$x(t) = \Sigma D_n e^{j\omega_0 nt} \longrightarrow \boxed{\begin{array}{c} 1 \uparrow h(t) \\ \\ -1 \quad\quad 1 \end{array}} \longrightarrow y(t) = \Sigma D_n H(j\omega_0 n) e^{j\omega_0 nt}$$

To Find output of an LTI system to a periodic input $x(t)$:

1- Find D_n coefficients of Fourier series of $x(t)$.

$$x(t) = \sum D_n e^{j\omega_0 nt}$$

$$D_n = \begin{cases} 1 & n = 0 \\ \dfrac{j2(-1)^n}{\pi n} & n \neq 0 \end{cases}$$

2- For ω_0, $2\omega_0$, $3\omega_0, ...,$ $n\omega_0$ (where ω_0 is the fundamental freq. of $x(t)$) find $H(j\omega_0 n)$:

$$H(j\omega_0 n) = \int_{-\infty}^{\infty} e^{-j\omega_0 n\tau} h(\tau) d\tau$$
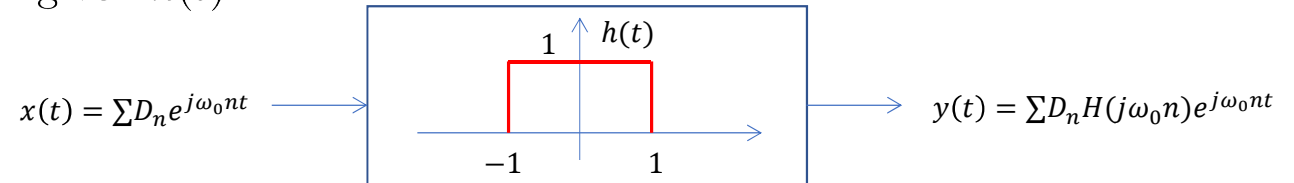
3- Coefficients of $y(t)$ (output of the system to $x(t)$) are $D_n H(j\omega_0 n)$ and the output is as follows:

$$y(t) = \sum D_n H(j\omega_0 n) e^{j\omega_0 nt}$$

Periodic Signals and LTI Systems

Passing $x(t)$ through LTI system with given $h(t)$.

$$x(t) = \Sigma D_n e^{j\omega_0 nt}$$

$$y(t) = \Sigma D_n H(j\omega_0 n) e^{j\omega_0 nt}$$
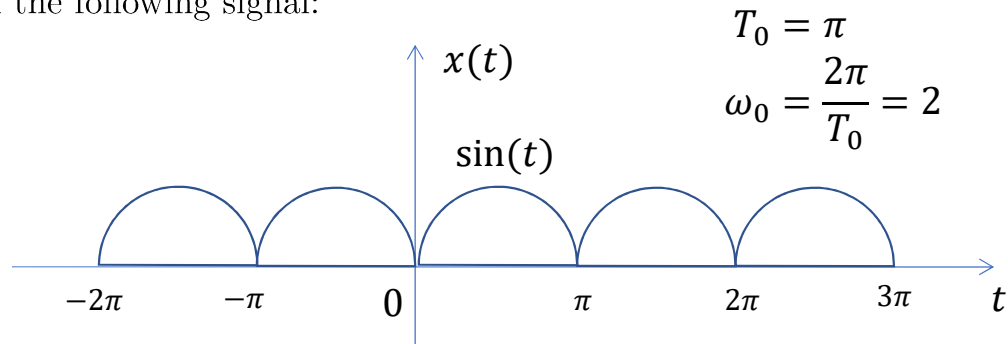
For the given signal $\omega_0 = \frac{\pi}{2}$

$$H(j\frac{\pi}{2}n) = \int_{-\infty}^{\infty} h(t) e^{-j\frac{\pi}{2}nt} dt = \int_{-1}^{1} 1 e^{-j\frac{\pi}{2}nt} dt$$

$$= \frac{e^{-j\frac{\pi}{2}nt}}{-j\frac{\pi}{2}n}\Big|_{-1}^{1} = \frac{e^{-j\frac{\pi}{2}n} - e^{j\frac{\pi}{2}n}}{-j\frac{\pi}{2}n}$$

$$= \frac{2sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n} \leftarrow Sinc\ Structure$$

$$D_n = \begin{cases} 1 & n = 0 \\ \frac{j2(-1)^n}{\pi n} & n \neq 0 \end{cases}$$

$$y(t) = \sum \frac{2sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n} D_n e^{j\omega_0 nt}$$

$$= 2e^{j\omega_0.0.t} + \sum_{n=-\infty,\ n\neq 0}^{\infty} 2\frac{sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n} \frac{2j(-1)^n}{\pi n} e^{j\omega_0 nt}$$

Periodic Signals

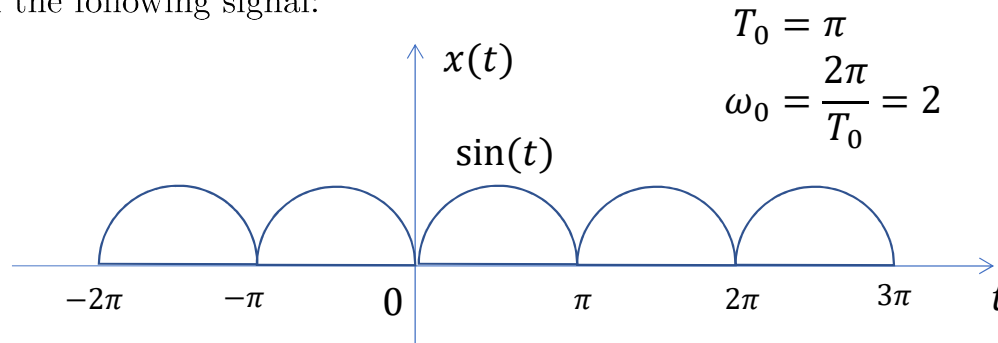**Example:** Find Fourier series of the following signal:

$$T_0 = \pi$$

$$\omega_0 = \frac{2\pi}{T_0} = 2$$

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt$$

Periodic Signals

Example: Find Fourier series of the following signal:

$$T_0 = \pi$$

$$\omega_0 = \frac{2\pi}{T_0} = 2$$

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt = \frac{1}{\pi} \int_0^\pi \sin(t) e^{-j2nt} dt$$

$$= \frac{1}{\pi} \int_0^\pi \left(\frac{1}{2j} e^{jt} - \frac{1}{2j} e^{-jt} \right) e^{-j2nt} dt$$

$$= \frac{1}{\pi} \int_0^\pi \frac{1}{2j} e^{j(1-2n)t} dt - \frac{1}{\pi} \int_0^\pi \frac{1}{2j} e^{-j(1+2n)t} dt$$

$$= \frac{1}{\pi} \frac{1}{2j} \frac{e^{j(1-2n)t}}{j(1-2n)} \Big|_0^\pi - \frac{1}{\pi} \frac{1}{2j} \frac{e^{-j(1+2n)t}}{-j(1+2n)} \Big|_0^\pi$$

Reminder:

$$e^{j\pi} = -1$$

$$e^{j\pi(1-2n)} = e^{j\pi} e^{-j2n\pi} = -1 \times 1$$

$$= \frac{1}{\pi} \left[\frac{e^{j\pi(1-2n)}}{2(-1)(1-2n)} - \frac{1}{2(-1)(1-2n)} \right] - \frac{1}{\pi} \left[\frac{e^{-j\pi(1+2n)}}{2(1+2n)} - \frac{1}{2(1+2n)} \right]$$

$$= \frac{2}{\pi} \frac{1}{1-2n} + \frac{2}{\pi} \frac{1}{1+2n}$$

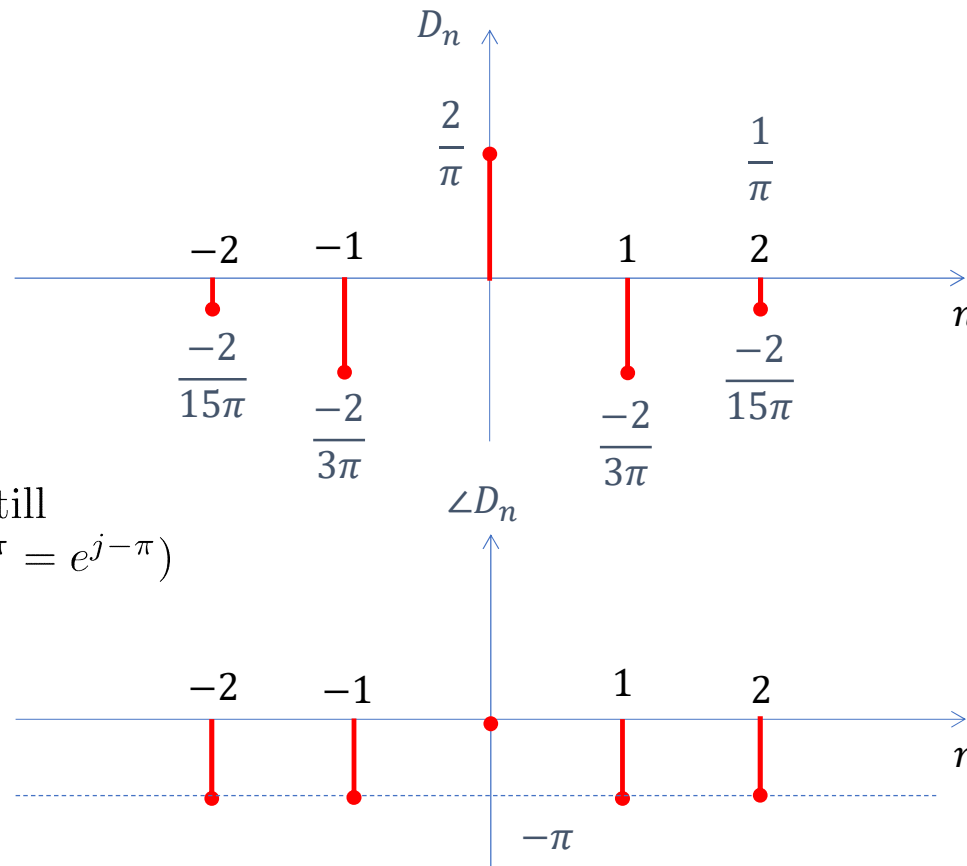$$= \frac{2(1+2n) + 2(1-2n)}{2\pi(1-4n^2)} = \frac{2}{\pi(1-4n^2)}$$

Plot $|D_n|$ and $\angle(D_n)$
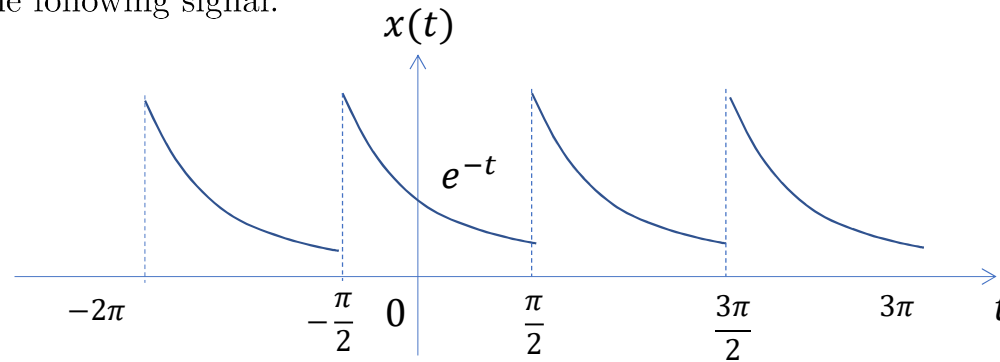
Periodic Signals

$$D_n = \frac{2}{\pi(1 - 4n^2)}$$

Note that this phase is still
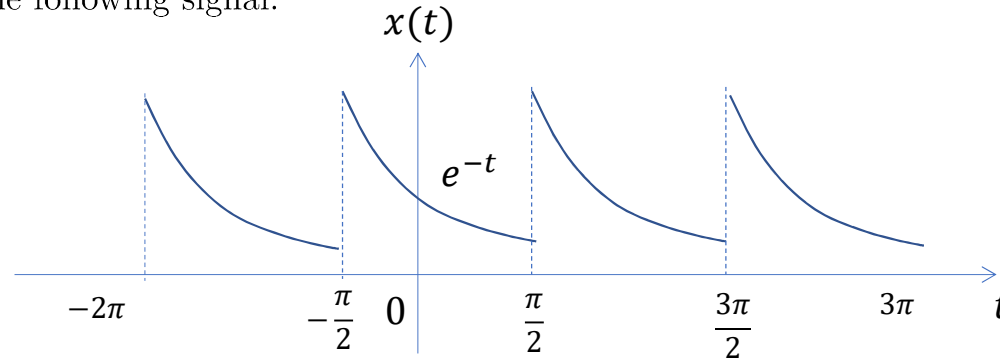an odd function of n $(e^{j\pi} = e^{j-\pi})$

Periodic Signals

Example: Find the Fourier series of the following signal:

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt$$

Periodic Signals

Example: Find the Fourier series of the following signal:
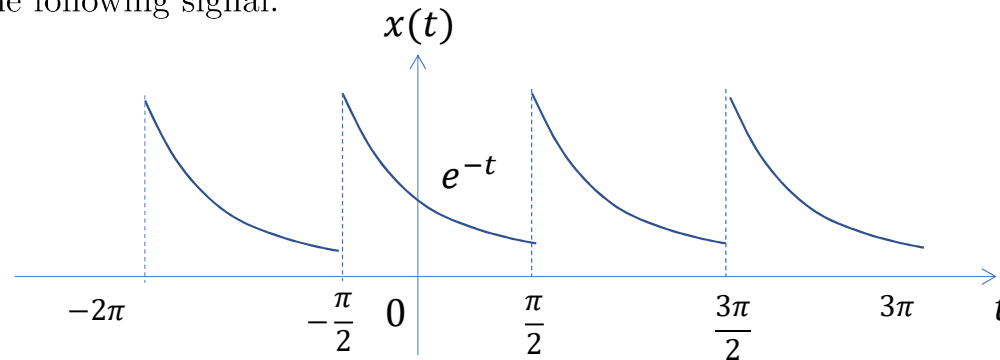

$$\omega_0 = \frac{2\pi}{\pi} = 2$$

$$T_0 = \pi$$

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{-t} e^{-j2nt} dt$$

$$= \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{-t(j2n+1)} dt = \frac{1}{-\pi(2nj+1)} \left(e^{-(j2n+1)\frac{\pi}{2}} - e^{(j2n+1)\frac{\pi}{2}} \right)$$

$$= \frac{1}{-\pi(2nj+1)} \left[e^{-jn\pi} e^{-\frac{\pi}{2}} - e^{jn\pi} e^{\frac{\pi}{2}} \right]$$

$$= \frac{1}{-\pi(2nj+1)} \left((-1)^n e^{-\frac{\pi}{2}} - (-1)^n e^{\frac{\pi}{2}} \right)$$

What are $|D_n|$s and $\angle(D_n)$?

$$= \frac{1}{\pi(2nj+1)} (-1)^n \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}} \right)$$

Periodic Signals

Example: Find the Fourier series of the following signal:
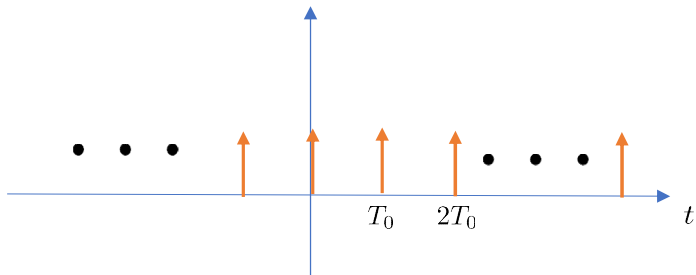

$$\omega_0 = \frac{2\pi}{\pi} = 2$$

$$T_0 = \pi$$

$$|D_n| = |\frac{1}{\pi(2nj+1)}(-1)^n \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)|$$

$$= |\frac{1}{\pi(2nj+1)}||(-1)^n|| \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)|$$

$$= \frac{1}{\sqrt{(2n\pi)^2 + \pi^2}} \times 1 \times \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)$$

$$\angle(D_n) = \angle(\frac{1}{\pi(2nj+1)}(-1)^n \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right))$$

$$= \angle(\frac{1}{\pi(2nj+1)}) + \angle((-1)^n) + \angle(\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right))$$

$$= \angle 1 - \angle(\pi(2nj+1)) + \angle((-1)^n) + 0$$

$$= \begin{cases} 0 - \tan^{-1}(2n) + 0 & n \ even \\ 0 - \tan^{-1}(2n) + \pi & n \ odd \end{cases}$$
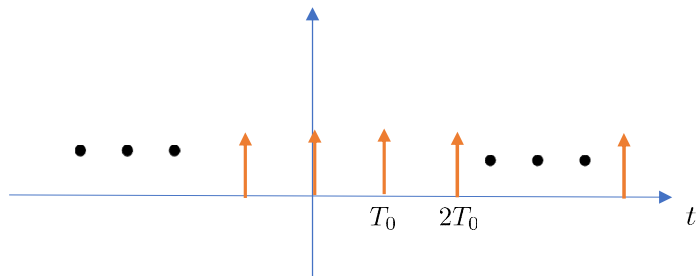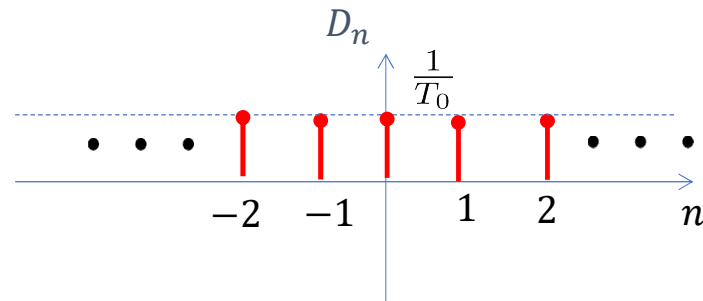
Periodic Signals

Example: Find FS of impulse train

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt$$
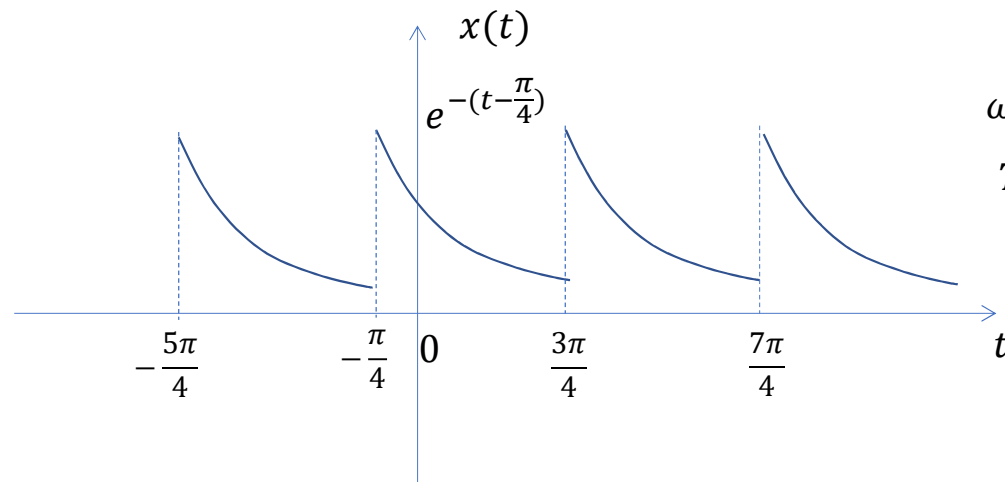
Periodic Signals

Example: Find FS of impulse train

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-j\omega_0 n 0} dt = \frac{1}{T_0}$$

Important signal in sampling

Periodic Input Delay, and LTI Systems

Find the fourier series of the following signal which is shifted version of the previous example by $\frac{\pi}{4}$
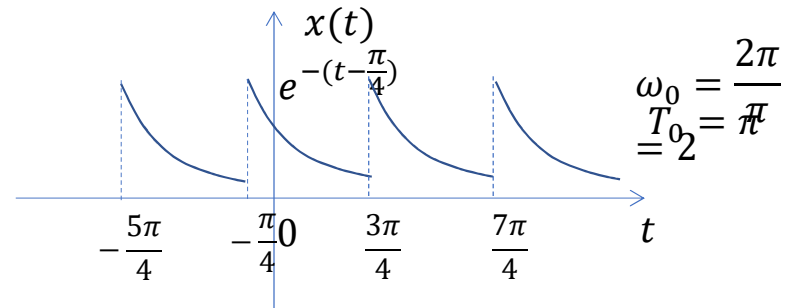

$$x(t)$$

$$e^{-(t-\frac{\pi}{4})}$$

$$\omega_0 = \frac{2\pi}{\pi} = 2$$

$$T_0 = \pi$$

$$-\frac{5\pi}{4} \qquad -\frac{\pi}{4} \quad 0 \qquad \frac{3\pi}{4} \qquad \frac{7\pi}{4} \qquad t$$

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt$$

Periodic Input Delay, and LTI Systems

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt = \frac{1}{\pi} \int_{\frac{-\pi}{4}}^{\frac{3\pi}{4}} e^{-(t-\frac{\pi}{4})} e^{-j2nt} dt$$

$$= \frac{e^{\frac{\pi}{4}}}{\pi} \int_{\frac{-\pi}{4}}^{\frac{3\pi}{4}} e^{-t(j2n+1)} dt = \frac{1}{-\pi(j2n+1)} e^{-(j2n+1)t} \Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{e^{\frac{\pi}{4}}}{-\pi(j2n+1)} \left(e^{-j2n.\frac{3\pi}{4}} e^{\frac{-3\pi}{4}} - e^{j2n.\frac{\pi}{4}} e^{\frac{\pi}{4}} \right)$$

$$= \frac{1}{-\pi(j2n+1)} \left(e^{-j2n.\frac{3\pi}{4}} e^{\frac{-\pi}{2}} - e^{j2n.\frac{\pi}{4}} e^{\frac{\pi}{2}} \right)$$

$$= \frac{1}{-\pi(j2n+1)} \frac{e^{j\frac{\pi}{2}n}}{e^{j\frac{\pi}{2}n}} \left(e^{-j2n.\frac{3\pi}{4}} e^{\frac{-\pi}{2}} - e^{j2n.\frac{\pi}{4}} e^{\frac{\pi}{2}} \right)$$

$$= \frac{1}{-\pi(j2n+1)} \frac{1}{e^{j\frac{\pi}{4}2n}} \left(e^{-j\pi n} e^{\frac{-\pi}{2}} - e^{j\pi n} e^{\frac{\pi}{2}} \right)$$

$$= e^{-j\frac{\pi}{4}2n} \underbrace{\frac{1}{-\pi(j2n+1)} \left(e^{-j\pi n} e^{\frac{-\pi}{2}} - e^{j\pi n} e^{\frac{\pi}{2}} \right)}_{\text{FS. of previous example}}$$

$x(t)$

$e^{-(t-\frac{\pi}{4})}$

$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2$

$-\frac{5\pi}{4}$ $-\frac{\pi}{4}$ 0 $\frac{3\pi}{4}$ $\frac{7\pi}{4}$ t

In case of delay the following is mostly useful:

$$e^{ja} + e^{jb} = e^{j\frac{a+b}{2}} \left(e^{j\frac{a-b}{2}} + e^{j\frac{-(a-b)}{2}} \right)$$
$$= e^{j\frac{a+b}{2}} \times 2\cos\left(\frac{a-b}{2}\right)$$

If $x(t) \xrightarrow{\text{Fourier series}} D_n$
then $x(t-t_0) \xrightarrow{\text{Fourier series}} D_n . e^{-j\omega_0 t_0 n}$

Some Properties of FS

Time Shift

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x(t - t_0) \xrightarrow{\text{Fourier series}} D_n . e^{-j\omega_0 t_0 n}$$

Linearity: If $x(t)$ and $y(t)$ have same fundamental Freqs.

$$x(t) \xrightarrow{\text{Fourier series}} D_n^x$$

$$y(t) \xrightarrow{\text{Fourier series}} D_n^y$$

$$Ax(t) + By(t) \xrightarrow{\text{Fourier series}} AD_n^x + BD_n^y$$

Product in time (convolution in Freq)
If $x(t)$ and $y(t)$ have same fundamental Freqs.

$$x(t) \xrightarrow{\text{Fourier series}} D_n^x$$

$$y(t) \xrightarrow{\text{Fourier series}} D_n^y$$

$$x(t)y(t) \xrightarrow{\text{Fourier series}} D_n^x * D_n^y$$

Scalar Multiplication

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$Ax(t) \xrightarrow{\text{Fourier series}} AD_n$$

Time Reversal

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x(-t) \xrightarrow{\text{Fourier series}} D_{-n}$$

Conjugation in time

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x^*(t) \xrightarrow{\text{Fourier series}} D_{-n}^*$$

Since $(e^{j\theta})^* = e^{-j\theta}$

Some Properties of FS

Time Scaling ($\alpha > 0$)

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x(\alpha t) \xrightarrow{\text{Fourier series}} D_n$$

Only fundamental freq. is changed from ω_0 to $\alpha\omega_0$

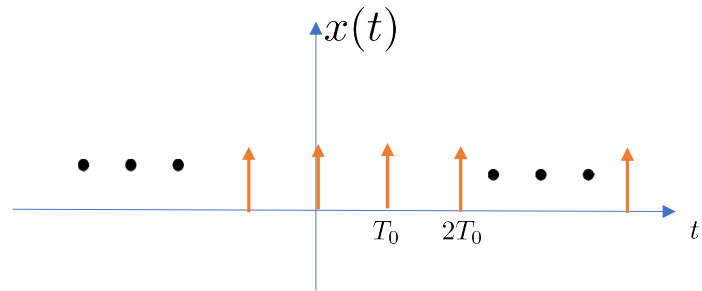$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt}$$

Fundamental freq is ω_0

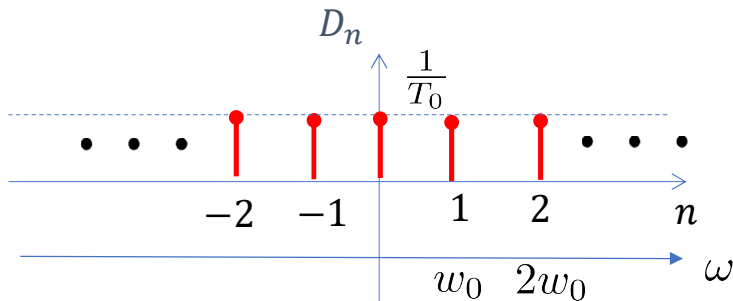$$x(\alpha t) = \sum_{n=-\infty}^{\infty} D_n e^{j(\alpha\omega_0)nt}$$

Fundamental freq is $\alpha\omega_0$ for $\alpha > 0$)

For $\alpha < 0$ the fundamental frequency is $|\alpha|\omega_0$ and the D_ns are therefore D_{-n}s
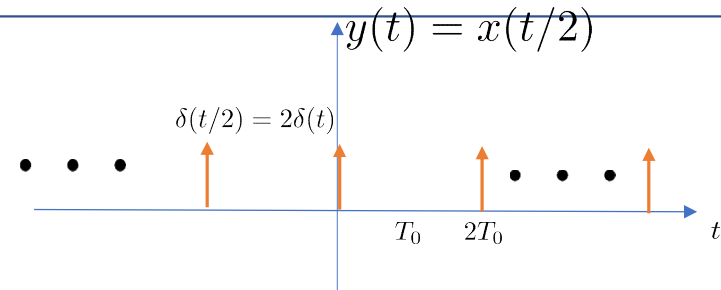
Time Scaling and FS

$x(t)$

$T_0 \quad 2T_0$

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-j\omega_0 n 0} dt = \frac{1}{T_0}$$

D_n

$\frac{1}{T_0}$

$-2 \quad -1 \qquad 1 \qquad 2 \qquad n$

$w_0 \quad 2w_0 \qquad \omega$

Time Scaling

$x(t) \xrightarrow{\text{Fourier series}} D_n \qquad$ Fundamental freq is ω_0

$x(\alpha t) \xrightarrow{\text{Fourier series}} D_n \qquad$ Fundamental freq is $\alpha \omega_0$

$y(t) = x(t/2)$

$\delta(t/2) = 2\delta(t)$

$T_0 \quad 2T_0 \qquad t$

Time Scaling and FS

$x(t)$

T_0 $2T_0$ t

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)e^{-j\omega_0 n0}dt = \frac{1}{T_0}$$

D_n

$\frac{1}{T_0}$

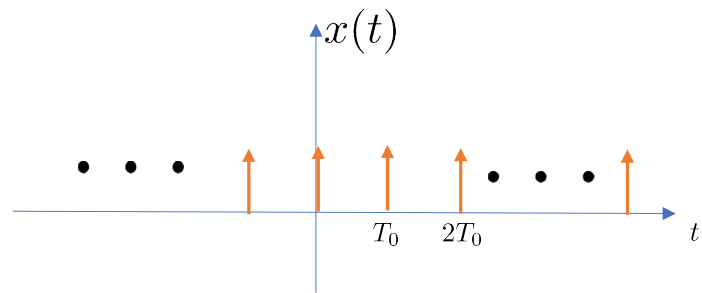-2 -1 1 2 n

w_0 $2w_0$ ω

Time Scaling

$$x(t) \xrightarrow{\text{Fourier series}} D_n \qquad \text{Fundamental freq is } \omega_0$$

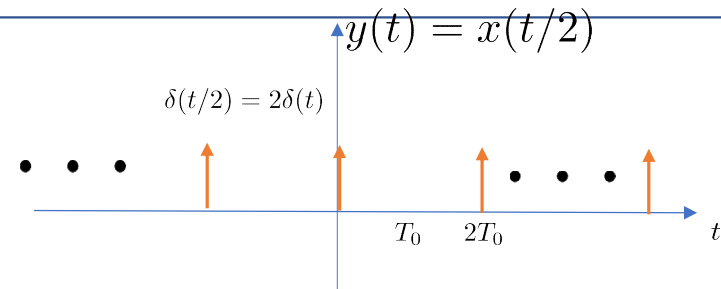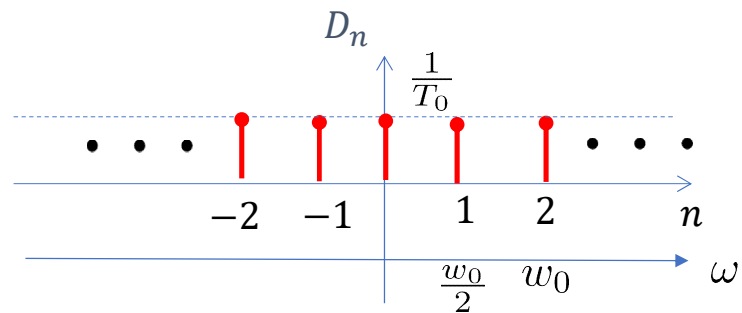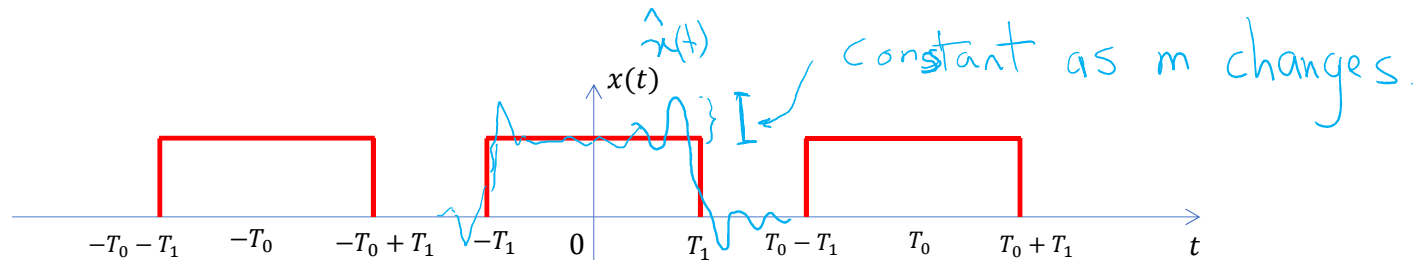$$x(\alpha t) \xrightarrow{\text{Fourier series}} D_n \qquad \text{Fundamental freq is } \alpha\omega_0$$

$y(t) = x(t/2)$

$\delta(t/2) = 2\delta(t)$

T_0 $2T_0$ t

$$D_n = \frac{1}{2T_0} \int_{<2T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{2T_0} \int_{-T_0}^{T_0} 2\delta(t)e^{-j\omega_0 n0}dt = \frac{1}{T_0}$$

D_n

$\frac{1}{T_0}$

-2 -1 1 2 n

$\frac{w_0}{2}$ w_0 ω

Using Finite Number of Coefficients (Gibbs Phenomenon)

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt}$$

$$\hat{x}(t) = \sum_{n=-m}^{m} D_n e^{j\omega_0 nt}$$

Windowing D_ns in frequency domain is equivalent to convolving with a sinc function in the time domain that results in the Gibbs Phenomenon.