Some results on structured $M$-matrices with an application to wireless communications

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Abstract

In this paper we study conditions under which a specially structured $Z$-matrix is an $M$-matrix. We apply the result to a capacity problem in wireless communications.

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1. Introduction

An $n \times n$ real matrix is called a $Z$-matrix if all of its off-diagonal entries are nonpositive. A $Z$-matrix is an $M$-matrix if it is nonsingular and its inverse is a nonnegative matrix, that is, all entries of the inverse are nonnegative. $M$-matrices are often used in applied fields. For example, to solve the optimal power control problem in wireless communications [4], one needs to solve the inequality $Ax \geq b$, where $A$ is a $Z$-matrix and $b$ is a positive vector. From the theory of $M$-matrices [1], the inequality $Ax \geq b$ has a positive solution if and only if $A$ is an $M$-matrix.

Motivated by engineering applications, in this paper we consider the following problem: given an $M$-matrix $A$ and nonnegative vectors $u_i$, $v_i$ for $i = 1, \ldots, k$, what are equivalent conditions for the perturbation $A - \sum_{i=1}^k u_i v_i^T$ of the $M$-matrix $A$ to be again an $M$-matrix?
A similar $M$-matrix perturbation situation arises in the study of ultrametric matrices. Historically, due to the fact that the inverse of a nonsingular nonnegative matrix is not an $M$-matrix in general, the so-called inverse $M$-matrix problem (see [2] for a survey) was first studied in [5], where a sufficient condition for a nonnegative symmetric matrix to be the inverse of a Stieltjes matrix (i.e., a nonsingular symmetric $M$-matrix) was established. This was achieved by introducing a class of matrices of type D which are inverses of Stieltjes matrices. Then Varga and Nabben [10] introduced the class of symmetric ultrametric matrices and proved that the inverse of a symmetric ultrametric matrix is a row and column diagonally dominant Stieltjes matrix. Moreover, the class of matrices of type D is contained in the class of symmetric ultrametric matrices. In 1994, Martínez et al. [6] studied the new class of (symmetric) strictly ultrametric matrices and stated that the inverse of a strictly ultrametric matrix is a strictly row and strictly column diagonally dominant Stieltjes matrix. A short proof of the main theorem in [6] is given in [7], where the class of strictly ultrametric matrices is characterized by using rank-one updates. The results in [6] were further generalized in [8] via the introduction of (nonsymmetric) generalized ultrametric matrices. There it is shown that the inverse of a generalized ultrametric matrix is a row and column diagonally dominant $M$-matrix, and a nonnegative matrix is a generalized ultrametric matrix if and only the matrix is a certain sum of at most rank-two matrices. Similar results were also obtained in [3] in which some nice determinantal inequalities were also obtained.

In Section 2 we study our $M$-matrix perturbation problem and apply our results to a capacity problem in wireless communications in Section 3.

2. Analysis of $A - \sum_{i=1}^{k} u_i v_i^T$

Let $A$ be an $n \times n$ $M$-matrix, and let $u_1, \ldots, u_k, v_1, \ldots, v_k$ be $n$-dimensional nonnegative column vectors. We study necessary and sufficient conditions for the $Z$-matrix

$$A - \sum_{i=1}^{k} u_i v_i^T$$

to be nonsingular with a nonnegative inverse, that is, to be an $M$-matrix. In this paper we need the following well known result on $M$-matrices [1].

**Lemma 2.1.** Let $A$ be an $n \times n$ $Z$-matrix. Then $A$ is an $M$-matrix if and only if all of its leading principal minors are positive.

**Lemma 2.2.** If $B, C$ are $n \times k$ matrices, then

$$\det(I_n + BC^T) = \det(I_k + C^T B),$$

where $I_n$ and $I_k$ are identity matrices of size $n$ and $k$, respectively.

**Proof.** It follows from the equality

$$\begin{bmatrix} I_n & 0_{n \times k} \\ C^T & I_k \end{bmatrix} \begin{bmatrix} I_n + BC^T & B \\ 0_{k \times n} & I_k \end{bmatrix} \begin{bmatrix} I_n & 0_{n \times k} \\ -C^T & I_k \end{bmatrix} = \begin{bmatrix} I_n & B \\ 0_{k \times n} & I_k + C^T B \end{bmatrix}. \quad \square$$

**Corollary 2.1.** If $A$ is an $n \times n$ nonsingular matrix and $B, C$ are $n \times k$ matrices, then

$$\det(A + BC^T) = \det A \cdot \det(I_k + C^T A^{-1} B).$$
Denote $U_i = [u_1, \ldots, u_i]$ and $V_i = [v_1, \ldots, v_i]$ for $i = 1, \ldots, k$. Then $A - \sum_{i=1}^{k} u_i v_i^T$ can be written as $A - U_k V_k^T$. Our first result shows that the problem of determining whether a certain $n \times n$ matrix is an $M$-matrix can be reduced to the problem whether a $k \times k$ matrix is an $M$-matrix.

**Theorem 2.1.** Suppose $A$ is an $n \times n$ $M$-matrix and $U_k, V_k$ are $n \times k$ nonnegative matrices. Then the $Z$-matrix $A - U_k V_k^T$ is an $M$-matrix if and only if the $k \times k$ matrix $I_k - V_k^T A^{-1} U_k$ is an $M$-matrix or equivalently,

$$\det(i - V_i^T A^{-1} U_i) > 0, \quad i = 1, \ldots, k.$$  

**Proof.** If $I_k - V_k^T A^{-1} U_k$ is an $M$-matrix, then $(I_k - V_k^T A^{-1} U_k)^{-1}$ exists and is nonnegative. The Sherman–Morrison–Woodbury formula [9] implies that

$$(A - U_k V_k^T)^{-1} = A^{-1} + A^{-1} U_k (I_k - V_k^T A^{-1} U_k)^{-1} V_k^T A^{-1}.$$  

Since $A^{-1}$ is nonnegative, so is $(A - U_k V_k^T)^{-1}$ from (4). Thus $A - U_k V_k^T$ is an $M$-matrix.

Now suppose $A - U_k V_k^T$ is an $M$-matrix. Since

$$A - U_k V_k^T \leq A - U_i V_i^T, \quad i = 1, \ldots, k,$$

it follows by [9, Theorem 2.4.10] that $A - U_i V_i^T$ is an $M$-matrix for $i = 1, \ldots, k$. From Corollary 2.1,

$$\det(A - U_i V_i^T) = \det(A) \cdot \det(i - V_i^T A^{-1} U_i),$$

and it follows that $\det(i - V_i^T A^{-1} U_i) > 0$. This shows that all leading principal minors of $I_k - V_k^T A^{-1} U_k$ are positive, which implies that $I_k - V_k^T A^{-1} U_k$ is an $M$-matrix by Lemma 2.1. □

The special case $k = 1$ below is useful not only for our next theorem, but also has applications in the next section.

**Corollary 2.2.** Let $A$ be an $n \times n$ $M$-matrix and let $u, v$ be $n$-dimensional nonnegative column vectors. Then the $Z$-matrix $A - uv^T$ is an $M$-matrix if and only if $v^T A^{-1} u < 1$.

**Theorem 2.2.** Suppose $A$ is an $n \times n$ $M$-matrix and $U_k, V_k$ are $n \times k$ nonnegative matrices. Then the $Z$-matrix $A - U_k V_k^T$ is an $M$-matrix if and only if for $i = 1, \ldots, k$, $(A - U_{i-1} V_{i-1}^T)^{-1}$ exists and

$$\delta_i \equiv 1 - v_i^T (A - U_{i-1} V_{i-1})^{-1} u_i > 0,$$

where $A - U_0 V_0^T$ is understood to be $A$.

**Proof.** Let $A - U_k V_k^T$ be an $M$-matrix. Then $A - U_{i-1} V_{i-1}^T$ is an $M$-matrix for $i = 1, \ldots, k$. Write $A - U_i V_i^T = (A - U_{i-1} V_{i-1}^T) - u_i v_i^T$, and apply Corollary 2.2, (5) follows.

Suppose (5) is satisfied. Since $A$ is an $M$-matrix and $\delta_1 > 0$, Corollary 2.2 implies that $A - U_1 V_1^T$ is an $M$-matrix. Repeatedly use Corollary 2.2, we see that $A - U_i V_i^T$ are all $M$-matrices for $i = 1, \ldots, k$. In particular, $A - U_k V_k^T$ is an $M$-matrix. □

Now we present a third equivalent condition for $A - U_k V_k^T$ to be an $M$-matrix. First, a lemma is needed.
Lemma 2.3. Let \( A \) be an \( m \times m \) \( \mathbb{Z} \)-matrix, \( g \) and \( f \) be \( m \)-dimensional nonpositive vectors, and \( \xi \) a positive number. Then the \( (m+1) \times (m+1) \) \( \mathbb{Z} \)-matrix

\[
\overline{A} = \begin{bmatrix} A & g \\ f^T & \xi \end{bmatrix}
\]

is an \( M \)-matrix if and only if \( A \) is an \( M \)-matrix and

\[
\xi - f^T A^{-1} g > 0.
\]

Proof. From Lemma 2.1, \( \overline{A} \) is an \( M \)-matrix if and only if \( A \) is an \( M \)-matrix and \( \det \overline{A} > 0 \). Since

\[
\overline{A} = \begin{bmatrix} I_m & 0 \\ f^T A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & g \\ 0^T & \xi - f^T A^{-1} g \end{bmatrix},
\]

we have

\[
\det \overline{A} = \det A \cdot (\xi - f^T A^{-1} g).
\]

Thus, the above equality implies that \( \overline{A} \) is an \( M \)-matrix if and only if \( A \) is an \( M \)-matrix and \( \xi - f^T A^{-1} g > 0 \). □

Theorem 2.3. Suppose \( A \) is an \( n \times n \) \( M \)-matrix and \( U_k, V_k \) are \( n \times k \) nonnegative matrices. Then \( A - U_k V_k^T \) is an \( M \)-matrix if and only if

\[
v_i^T A^{-1} u_i < 1 - v_i^T A^{-1} (I_i - V_i^T A^{-1} U_i)^{-1} V_i^T A^{-1} u_i, \quad i = 1, \ldots, k,
\]

where (7) is understood as \( v_1^T A^{-1} u_1 < 1 \) when \( i = 1 \).

Proof. By Theorem 2.1, for \( i = 1, \ldots, k, A - U_i V_i^T \) is an \( M \)-matrix if and only if \( I_i - V_i^T A^{-1} U_i \) is an \( M \)-matrix. Applying Lemma 2.3 to the \( Z \)-matrix

\[
I_i - V_i^T A^{-1} U_i = \begin{bmatrix} I_i & -V_i^T A^{-1} U_i \\ -v_i^T A^{-1} U_i & 1 - v_i^T A^{-1} u_i \end{bmatrix}, \quad i = 2, \ldots,
\]

the theorem follows. □

As an application of this section’s results, we characterize the positive solution of a particular inequality.

Theorem 2.4. Suppose \( A \) is an \( n \times n \) \( M \)-matrix and \( U_k, V_k \) are \( n \times k \) nonnegative matrices such that \( V_k \neq 0 \). Let \( b \) be an \( n \)-dimensional positive column vector. Then the inequality

\[
(A - U_k V_k^T)x \geq b
\]

has a positive solution if and only if \( A - U_k V_k^T \) is an \( M \)-matrix. In this case,

\[
x^* = (A - U_k V_k^T)^{-1} b
\]

is the minimal solution of (8), i.e., any solution \( x \) to (8) satisfies \( x \geq x^* \).

Proof. If \( A - U_k V_k^T \) is an \( M \)-matrix, then \( x^* = (A - U_k V_k^T)^{-1} b \) is a positive solution of (8) since \( (A - U_k V_k^T)x^* = b \). Conversely, let \( x \) be a positive solution of (8). Then \( (A - U_k V_k^T)x \geq b \) implies that
\((I_k - V_k^T A^{-1} U_k)V_k^T x \geq V_k^T A^{-1} b.\)

Denote \(P = V_k^T A^{-1} U_k, y = V_k^T x,\) and \(d = V_k^T A^{-1} b.\) Then we have \((I_k - P)y \geq d,\) from which it follows that

\[
\begin{align*}
y & \geq Py + d = P( Py + d ) + d = P^2 y + Pd + d \\
& \geq P^2( Py + d ) + Pd + d = P^3 y + P^2 d + Pd + d \\
& \geq \ldots \geq P^{m+1} y + \sum_{i=0}^{m} P^i d \geq \sum_{i=0}^{m} P^i d
\end{align*}
\]

for any positive integer \(m.\) Thus the series \(\sum_{i=0}^{\infty} P^i d\) converges. Since \(b\) is a positive vector and \(A^{-1}\) is nonnegative, \(A^{-1} b\) is positive. So the assumption that \(V_k \neq 0\) implies that \(d\) is a positive vector. It follows that \(\sum_{i=0}^{\infty} P^i z\) converges for any \(n\)-dimensional vector \(z.\) Therefore

\[
(I - P)^{-1} = \sum_{i=0}^{\infty} P^i
\]

exists and is nonnegative. Thus \(I_k - V_k^T A^{-1} U_k\) is an \(M\)-matrix, and so is \(A - U_k V_k^T\) by Theorem 2.1. The last conclusion is obvious. \(\square\)

3. An application

In the third generation wireless communication systems, the power and data rate allocation problem is one of the bottlenecks of the feasibility of multiclass services. This problem can be reduced to solving an inequality system, which is constructed by service requirements and service restrictions [4,11].

As an application of the above results, we consider a code-division multiple-access (CDMA) communication system, supporting multi-class services. The \(i\)th service class is specified by its target signal energy to interference power spectral density ratio (SIR), \(\gamma_i,\) and processing gain, \(G_i,\) which is the ratio of chip rate to the data rate. We further assume that the number of service classes supported is \(K,\) and the number of active users in the \(i\)th service class is \(n_i.\) The received SIR for the \(i\)th service class can be expressed as

\[
\text{SIR}_i = \frac{S_i G_i}{\sum_{k=1}^{K} n_k S_k - S_i + \sigma_i^2}, \quad (10)
\]

where \(S_i\) and \(\sigma_i^2\) are respectively the allocated power and background noise seen by an \(i\)th class user. The noise component may include AWGN (additive white Gaussian noise) and inter-cell interference. \(n_k\) is the number of users in the \(k\)th class. The numerator of \((10)\) represents the power for the desired use multiplied by the processing gain. The denominator is the total interference, including power sum from all other users in the system and the noise component. Satisfactory system operation requires that

\[
\text{SIR}_i \geq \gamma_i, \quad (11)
\]

where \(\gamma_i\) is the target SIR for the \(i\)th class user, which is specified by the target bit error rate, modulation, and coding schemes. By manipulating \((10)\) and \((11),\) we can obtain a set of linear inequality requirements as
\( S_i \left( 1 + \frac{G_i}{\gamma_i} \right) \geq \sum_{k=1}^{K} n_k S_k + \sigma_i^2, \quad 1 \leq i \leq K. \) \hspace{1cm} (12)

Let \( d_i = 1 + \frac{G_i}{\gamma_i} \) and \( b_i = \sigma_i^2 \), for \( i = 1, \ldots, K \). We define the power vector \( S = (S_1, S_2, \ldots, S_K)^T \) and the nonnegative noise vector \( b = (b_1, b_2, \ldots, b_K)^T \). If we let \( e \) be the \( K \)-dimensional vector of all 1’s, then we can write the inequality set (12) as

\[
(D - ev^T) \cdot S \geq b,
\]

where \( D = \text{diag}(d_1, \ldots, d_K) \) and \( v = (n_1, \ldots, n_K)^T \). The optimal power allocation problem is to find the power vector solution \( S^* \) such that any other solution of (13) satisfies \( S \geq S^* \). The capacity problem is to determine the \( K \)-dimensional region of \((n_1, n_2, \ldots, n_K)\) on which (13) is solvable. This region specifies the maximal allowable number of users that can be supported by the system. See [11] for more details. Corollary 2.1 immediately gives the following answer to the problem.

**Proposition 3.1.** The system (13) has a positive solution if and only if

\[
\sum_{i=1}^{K} \frac{n_i}{1 + \frac{G_i}{\gamma_i}} < 1.
\]

If (14) is satisfied, then the optimal power allocation is given by

\[
S^* = (D - ev^T)^{-1} b = \left( I + \frac{1}{\eta} D^{-1} ee^T \right) D^{-1} b
\]

where

\[
\eta \equiv 1 - \sum_{i=1}^{K} \frac{n_i}{1 + \frac{G_i}{\gamma_i}} > 0.
\]

The inequality (14) can be used to specify the \( K \)-dimensional region of the allowed numbers \((n_1, n_2, \ldots, n_K)\) of users in the service classes for the capacity problem of the communication system.

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